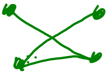

CSC165 fall 2017

graph connectivity, trees

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Using Course notes: average analysis; graphs

Outline

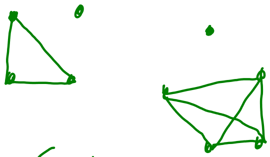
notes

must be connected

$\forall n \in \mathbb{N}, \exists M \in \mathbb{N}, \forall G = (V, E), (V = n \wedge E \geq M) \Rightarrow G$ is connected?

$\frac{n(n-1)}{2} = |E|$

$\frac{(n-1)(n-2)}{2}$



$\forall n \in \mathbb{N}^+, \forall G = (V, E), \left(|V| = n \wedge |E| \geq \frac{(n-1)(n-2)}{2} + 1\right) \Rightarrow G$ is connected

Proof (induction on n).

base case - vacuously true — no graph with 1 vertex and ≥ 1 edge

must be connected

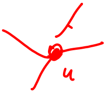
$\forall n \in \mathbb{N}, \exists M \in \mathbb{N}, \forall G = (V, E), (\ V\ = n \wedge\ E\ \geq M) \Rightarrow G$ is connected?

P(n) " " $\frac{(n-1)(n-2)}{2}+1$

Inductive step Let $k \in \mathbb{N}^+$. assume $P(k)$. Want to show $P(k+1)$ must follow.

Don't do this!!
Start with arb graph on k vertices with $\frac{(k-1)(k-2)}{2}+1$ edges. By IH it is connect. add $k-1$ edges + vertex, result connected. induction trap

Do this Let $G = (V, E)$ be an arbitrary graph with $|V| = k+1$ and $|E| = \frac{k(k-1)}{2} + 1$.



Trap II: choose $u \in V$, let $G' = (V', E')$ where $V' = V \setminus \{u\}$, $E' = E \setminus \{(u,v) \mid (u,v) \in E\}$

must be connected

$\forall n \in \mathbb{N}, \exists M \in \mathbb{N}, \forall G = (V, E), (\ V\ = n \wedge\ E\ \geq M) \Rightarrow G$ is connected?

Then $|V'| = k$ $\quad |E'| \geq \frac{k(k-1)}{2} + 1 - k$

$$= \frac{k(k-1)}{2} - \frac{2k}{2} + 1$$

$$= \frac{k(k-3)}{2} + 1 = \frac{k^2 - 3k}{2} + 1$$

$$< \frac{k^2 - 3k + 2}{2} + 1 = \frac{(k-1)(k-2)}{2} + 1$$

oop!

case 1 $|E| = \frac{k(k+1)}{2}$, so G is complete & thus connected

case 2 $|E| < \frac{k(k+1)}{2}$ Need vertex u with $1 \leq d(u) \leq k-1$

~~maybe~~ must be connected

$\forall n \in \mathbb{N}, \exists G = (V, E),\ V = n \wedge E = n - 1 \wedge G$ is connected

Then $d(u) > 0$. (otherwise $G-u$ has $\leq \frac{k(k-1)}{2}$ $\rightarrow\leftarrow$)

also, if G is not complete $\exists\, u, v \in V$ s.t. $(u,v) \notin E$

Choose u, note $d(u) \leq k-1$.

Let $G' = (V', E')$, $V' = V \setminus \{u\}$, $E' = E \setminus \{(u,v) \mid (u,v) \in E\}$

Then $|V'| = k$ and $|E'| \geq |E| - (k-1)$

$$\geq \frac{k(k-1)}{2} + 1 - (k-1) \qquad \text{IH}$$

$$= \frac{k(k-1)}{2} - \frac{2(k-1)}{2} + 1$$

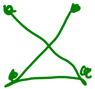
$$= \frac{(k-1)(k-2)}{2} + 1$$

Thus, by IH, G' is connected.

maybe be connected

$\forall n \in \mathbb{N}, \exists G = (V, E),\ V \ = n \wedge \ E \ = n - 1 \wedge G$ is connected

– P_n – path on n vertices – is connected by n-1 edges



Case study G = Facebook

$|V| = 2B$ (?)

$\frac{(2B-1)(2B-2)}{2} + 1 \rightarrow$ G connected

2B-1 friendships which friends if controlled

maybe be connected

$\forall n \in \mathbb{N}, \exists G = (V, E),\ V \ = n \wedge \ E \ = n - 1 \wedge G$ is connected

Since $d(u) \geq 1$, u has/had neighbour $w \in V'$

Thus u connected to w & w connected to all other vertices $\Rightarrow$ G connected $\blacksquare$

must be disconnected

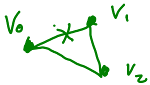
$\forall n \in \mathbb{N}, \forall G = (V, E), (\ V\ = n \land\ E\ \leq n - 2) \Rightarrow G$ is not connected

steps:

- natural to reason by removing an edge from a connected graph with $n \quad 1$ edges...
- first need some results about which components of connected graphs have redundant edges (cycles)...
- then need some results about connected graphs without cycles (trees)... 
- then reason about reducing an arbitrary connected graph to a tree... (by pruning)

whew!

cycle



consecutively adjacent vertices $v_0, \ldots, v_k \in V \wedge k \geq 3$,
all distinct except $v_0 = v_k$

$\forall G = (V, E), \forall e \in E, G$ connected $\Rightarrow$ (e in a cycle of $G \Leftrightarrow G - e$ connected)

Proof $\rightarrow$ Let $G=(V,E)$, let $e \in E$, assume G is connected. Assume e in a cycle.

Let $w_1, w_2 \in V$. Since G is connected, there is path, P, from w_1 to w_2.

Case 1 $e=(u,v)$ not in P. Then P is a path in $G-e$. w_1, w_2 are connected in $G-e$.

Case 2 Path P includes edge $e=(u,v)$. Let P_1 be ~~the~~ the path from w_1 to nearest of u,v in P. Let P_2 be the path from w_2 to nearest of u,v in P. Since $e=(u,v)$ is in a cycle, there is a path from u to v after $e(u,v)$ removed. call P_3

cycle

consecutively adjacent vertices $v_0, \ldots, v_k \in V \wedge k \geq 3$,
all distinct except $v_0 = v_k$

$\forall G = (V, E), \forall e \in E, G$ connected $\Rightarrow$ (e in a cycle of $G \Leftrightarrow G - e$ connected)

the $P_1 \cup P_2 \cup P_3$ is a path w_1 to w_2 in $G-e$.

Proof $\Leftarrow$ Idea if $e=(u,v)$ and $G-e$ connected

then $\exists$ Path from u to v in $G-e$.

Then show Path $\cup\,(u,v)$ is a cycle

tree: connected, acyclic graph

tree acyclic connected graph

removing any edge from a tree disconnects it

$\forall G = (V, E), G \text{ is a tree } \Rightarrow \; E \; = \; V \; - 1$

but first...

$\forall G = (V, E), (G \text{ is a tree } \wedge \; V \; \geq 2) \Rightarrow (\exists v \in V, d(v) = 1)$

reason want to let $G = (V, E)$ be an arbitrary tree $|V| = n+1$ vertices

- remove 1 vertex (and 1 edge)

on use IH

tree: connected, acyclic graph

removing any edge from a tree disconnects it

$\forall G = (V, E), G \text{ is a tree } \Rightarrow \; E \; = \; V \; - 1$

but first...

$\forall G = (V, E), (G \text{ is a tree } \wedge \; V \; \geq 2) \Rightarrow (\exists v \in V, d(v) = 1)$

main result...

$\forall n \in \mathbb{N}^+, \forall G = (V, E), (G \text{ is a tree } \wedge\ V\ = n) \Rightarrow\ E\ =\ V\ - 1$

$P(n)$:

Base case The only tree with $|V| = 1$ is graph with $0 = 1-1$ edges.

Inductive step Let $n \in \mathbb{N}^+$. Assume $P(k)$ $\forall k \in \mathbb{N}^+$, $k \le n$. i.e every tree with $k \le n$ vertices has exactly $k-1$ edges. Want to prove

$P(n+1)$: $\forall k \in \mathbb{N}$, $k \le n+1$, tree with k vertices has $k-1$ edges.

Let $T = (V, E)$ be a tree with $|V| = n+1$

Remove edge $e = (u, v)$ from E. T is disconnected

Let $V_1 \subseteq V$ be vertices that were connected to u without using v.

main result...

$\forall n \in \mathbb{N}^+, \forall G = (V, E), (G \text{ is a tree} \wedge \; V \; = n) \Rightarrow \; E \; = \; V \; - 1$

Let $V_2 = V \setminus V_1$. $V_1 \cap V_2 = \emptyset$ (due to no cycles).

and $V_1 \cup V_2 = V$.

$T_1 = (V_1, E_1)$ $T_2 = (V_2, E_2)$

$1 \le |V_1| \le n$; also $n \le |V_2| \le n$

$|E_1| = |V_1| - 1$ (IH) and $|E_2| = |V_2| - 1$

so $|E| = |E_2| + |E_1| + 1$

$= |V_1| - 1 + |V_2| - 1 + 1$

$= |V_1| + |V_2| - 1 = |V| - 1$ ∎

big picture...

must be disconnected | $n-1$ | $\frac{(n-1)(n-2)}{2}$ | must be connected

$|E| = 0$

$|E| = \frac{n(n-1)}{2}$

slightly alter parameters by specifying ~~maximum~~ minimum degree

Notes