CSC165 fall 2017

graph connectivity, trees

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Using Course notes: average analysis; graphs

Outline

notes

must be connected

$\forall n \in \mathbb{N}, \exists M \in \mathbb{N}, \forall G = (V, E), (|V| = n \wedge |E| \geq M) \Rightarrow G$ is connected?

$n-1$ — too small $\frac{n(n-1)}{2}$ — too big?

$\frac{(n-1)(n-2)}{2}$ — too small

Prove that $M = \frac{(n-1)(n-2)}{2} + 1$ will do

$\forall n, \forall G = (V, E) \left(|V| = n \wedge |E| \geq \frac{(n-1)(n-2)}{2} + 1 \right) \Rightarrow G$ is connected

base case $n = 1$ — vacuously true.

must be connected

$\forall n \in \mathbb{N}, \exists M \in \mathbb{N}, \forall G = (V, E), (\ V \ = n \wedge \ E \ \geq M) \Rightarrow G$ is connected?

$P(n): \forall G = (V,E), \left(|V| = n \wedge |E| \geq \frac{(n-1)(n-2)}{2} + 1\right) \Rightarrow G$ is connected.

Inductive step: Let $n \in \mathbb{N}$. Assume $P(n)$. Want to show $P(n+1)$ follows.

Wrong approach - don't do this
take a graph with n vertices + suitable number of edges + extend to graph with $n+1$ vertices - show it's connected
WRONG! - this only shows existence of an example with $n+1$ vertices

Let $G = (V, E)$ have $|V| = n+1$ and $|E| \geq \frac{n(n-1)}{2} + 1$ vertices.

must be connected

$\forall n \in \mathbb{N}, \exists M \in \mathbb{N}, \forall G = (V, E), (|V| = n \wedge |E| \geq M) \Rightarrow G$ is connected?

Let $G' = (V', E')$ where $V' = V \setminus \{u\}$ and

$E' = E \setminus \{(u,v) \mid (u,v) \in E\}$

want u to be connected to G'

and $\frac{n(n-1)}{2} + 1 - n \geq \frac{(n-1)(n-2)}{2} + 1$

show $\delta(u) = \delta(v) = 0$ impossible

problem algebra doesn't work

Fix if G is complete, i.e. all edges, then between any $u,v \in V$ there an edge, hence path

i.e. $|E| = \frac{(n+1)n}{2}$

Case G is not complete $\Rightarrow \exists u,v \in V$, $d(u), d(v) \leq n-1$

select $u \in V$ s.t. $1 \leq d(u) \leq n-1$

must be connected

$\forall n \in \mathbb{N}, \exists M \in \mathbb{N}, \forall G = (V, E), (V \ = n \wedge E \ \geq M) \Rightarrow G$ is connected?

Now $G' = (V', E')$ where $V' = V \setminus \{u\}$, $|V'| = n$
and $E' = E \setminus \{(u,v) \mid (u,v) \in E\}$

Then $|E'| \geq |E| - (n-1)$

$$= \frac{n(n-1)}{2} + 1 - (n-1)$$

$$= \frac{n(n-1) - 2n}{2} + 2 \geq \frac{(n-1)(n-2)}{2} + 1$$

$\Rightarrow G'$ connected (by IH)

also $\exists v \in V'$ s.t. $(u,v) \in E$, since $d(u) \geq 1$

$\therefore G$ is connected.