Learning Objectives

By the end of this worksheet, you will:

- Translate sentences between natural English and predicate logic.

- Determine the truth value of sentences on small domains.

- Define a property of a mathematical function using predicate logic.

- Express conditions on domains in predicate formulas.

1. **Alternating quantifiers.** Consider the following table that shows the employees of a (very small) company:[1]

Employee	Salary	Department
Aizah	60,000	Sales
Betty	15,000	Sales
Carlos	40,000	HR
Doug	30,000	Sales
Ellen	50,000	Design
Flo	20,000	Design

Let E be the set of the six employees listed above. We'll define two predicates on these sets:

$$Rich(x) : \text{``}x \text{ earns more than 35,000,''} \text{ where } x \in E$$
$$SameDept(x, y) : \text{``}x \text{ and } y \text{ are in the same department,''} \text{ where } x, y \in E$$

(a) Consider the statement
$$\exists x, y \in E, \ Rich(x) \wedge SameDept(x, y).$$

Give one *example* to show that this statement is *true*. Is there more than one possible answer? Note: just as in programming and math, two variables x and y can have the same value (i.e., refer to the same employee).

> **Solution**
>
> $x = $ Aizah and $y = $ Aizah is one possibility. Keep in mind that the variables don't need to be different elements of E!
>
> A good exercise is to try to find all possbilities.

(b) Here's the same statement, but with a universal quantifier instead:
$$\forall x, y \in E, \ Rich(x) \wedge SameDept(x, y).$$

Give one *counter-example* to show that this statement is *false*. (Is there more than one possible answer?)

> **Solution**
>
> Lots of possibilities here! We can pick an employee who isn't rich, e.g. $x = Betty$ and $y = Aizah$. Or, we could pick two employees who aren't in the same department, e.g. $x = Aizah$ and $y = Carlos$.

(c) Now let's look at alternating quantifiers.
$$\forall y \in E, \ \exists x \in E, \ Rich(x) \wedge SameDept(x, y).$$

Is this true or false? How do you know?

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[1] This example is adapted from an old set of CSC165 course notes.

> **Solution**
>
> This is true. The translation is "for every employee y, there is an employee x who has a salary greater than 30,000 and in the same department as y." Each department has an employee with a salary greater than 30,000.
>
> Note that HR only has one employee! So if $y = Carlos$, we would also pick $x = Carlos$ to make this statement true.

(d) With the quantifiers switched:

$$\exists x \in E, \ \forall y \in E, \ Rich(x) \wedge SameDept(x, y).$$

Explain why this statement is *false*.

> **Solution**
>
> This statement says that there is an employee x who has a salary greater than 30,000 and where every employee is in the same department as x. This is false, since there are employees in different departments.

2. **A property of functions**. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Recall that the *range* of f is the set after the $\to$ symbol (in this case, $\mathbb{R}$), which contains the possible output values of f. However, in general the range of a function might contain values that cannot possibly be output (e.g., if $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^2$).

We say that $f : \mathbb{R} \to \mathbb{R}$ is **onto** if and only if its range $\mathbb{R}$ *only* contains values that could possibly be output by f (and no "impossible" values). For example, $f(x) = x^2$ is not onto, but $f(x) = x + 1$ is onto.

(a) Express this definition as a predicate in symbolic form, filling in the following blank. You may use an expression like $f(x) = y$ in your formula.[2]

$Onto(f)$:   , where $f : \mathbb{R} \to \mathbb{R}$.

> **Solution**
>
> $\forall y \in \mathbb{R}, \ \exists x \in \mathbb{R}, \ f(x) = y$.

(b) Let $f(x) = x^2$. Give a *counter-example* to show that f is not onto.

> **Solution**
>
> Because x^2 is always greater than or equal to 0, if we pick $y = -1$, the statement $\exists x \in \mathbb{R}, \ x^2 = -1$ is false!

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[2] Recall that a legitimate formula is built up only from quantifiers (each ranging over a defined set), boolean operators ($\wedge$, $\vee$, $\neg$, etc.), and predefined functions and predicates.

Just as how when we write a^b, this is actually shorthand for an exponentiation function $exp(a, b)$, when we write $f(x)$ in a formula this is shorthand for a *function evaluation* operator $F(f, x)$. For the set A of all functions from $\mathbb{R}$ to $\mathbb{R}$, this operator is defined as $F : A \times \mathbb{R} \to \mathbb{R}$, where $F(f, x)$ is equal to $f(x)$. So while you can write a formula like $\forall x \in \mathbb{R}, \ f(x) = 10$, this is actually shorthand for the more technically correct $\forall x \in \mathbb{R}, \ F(f, x) = 10$.

3. **Expressing conditions.** Often when we want to express statements using predicate logic, the common domains like $\mathbb{N}$ and $\mathbb{R}$ do not quite fit. For example, consider the statement:

<div align="center">

Every natural number greater than 3 is greater than 1.

</div>

The use of "every" suggests a universal quantification, but using the domain of all natural numbers in the following way does not capture the full meaning of the statement:

$$\forall n \in \mathbb{N},\ n > 1$$

In this question, you'll explore two equivalent ways of modifying the above formula to capture the condition "greater than 3."

(a) One simple way is to change the domain over which we're quantifying. Write down the definition of a set S such that the following formula is equivalent to the original statement:

$$\forall n \in S,\ n > 1$$

> **Solution**
>
> $S = \{n \mid n \in \mathbb{N} \text{ and } n > 3\}$.

(b) We've hinted at another way to express this statement by using the word "condition" to describe the role of "greater than 3." The original statement only talks about natural numbers greater than 3, and ignores all others – the same is true of implication. This indicates that we can keep the original domain, but use an implication to narrow the scope of what we're talking about.

Define a predicate $P(n)$ such that the following formula is equivalent to the original statement:

$$\forall n \in \mathbb{N},\ P(n) \Rightarrow n > 1$$

> **Solution**
>
> $P(n) : n > 3$, where $n \in \mathbb{N}$.

(c) Using what you've learned, express the following statement in *two different* ways:

<div align="center">

Every integer that is greater than 10 or less than -40 is not equal to 0.

</div>

> **Solution**
>
> Using the domain $S = \{x \mid x \in \mathbb{Z} \text{ and } x > 10 \text{ or } x < -40\}$,
>
> $$\forall x \in S,\ x \neq 0$$
>
> Using the predicate $P(x) : \text{"}x > 10 \vee x < -40\text{"}$,
>
> $$\forall x \in \mathbb{Z},\ P(x) \Rightarrow x \neq 0$$
>
> or simply
>
> $$\forall x \in \mathbb{Z},\ (x > 10 \vee x < -40) \Rightarrow x \neq 0$$