CSC165 fall 2017

Mathematical expression: predicate logic

read here

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Using Course notes: Mathematical Expression: predicate logic

Outline

compare and contrast...

"If $\underbrace{\text{it rains}}_{r}$, then $\underbrace{\text{I will wear sneakers}}_{s}$." $r \Rightarrow s$

"If and only if it rains, then I will wear sneakers." $r \Leftrightarrow s$

$r \wedge s$ — $r \Rightarrow s \to T$ — $r \Leftrightarrow s$ T

$r \wedge \neg s$ — $r \Rightarrow s \to F$ — $r \Leftrightarrow s$ F

$\neg r \wedge s$ — $r \Rightarrow s \to T$ — $r \Leftrightarrow s$ F

$\neg r \wedge \neg s$ — $r \Rightarrow s \to T$ — $r \Leftrightarrow s$ T

$0 > 3 \Rightarrow 0 > 1$

what's a predicate?

boolean functions $\to \{T, F\}$

$n > 7.2$

x is tall

$n = 3 \to F$

$n = 15 \to T$

$n =$ "Fred" $\to$!??#!WTF!

$x =$ Danny $\to$ False

$x =$ KK $\to$ True

$x = 3 \to$

domain

$n > 7.2$: $\mathbb{R}$

x is tall: {objects with height}

predicate definitions

$IsTall(x): \text{Animals} \longrightarrow \{T, F\}$

where $IsTall(x)$ T iff x is > 3 metres

$G(n)$: "$n > 7.2$", where $n \in \mathbb{R}$

quantifiers ∀ and ∃

$n > 7.2$

$\forall n \in \mathbb{N}, n > 7.2$

means $0 > 7.2 \wedge 1 > 7.2 \wedge 2 > 7.2 \wedge 3 > 7.2 \wedge 4 > 7.2, \ldots$

False — counterexample $7 \not> 7.2$

$\exists n \in \mathbb{N}, n > 7.2$

means $0 > 7.2 \vee 1 > 7.2 \vee 2 > 7.2 \vee 3 > 7.2 \vee 4 > 7.2 \vee, \ldots$

True: example $8 > 7.2$

Special cases

$\forall x \in \emptyset, \text{blah}(x)$, always True!

$\exists x \in \emptyset, \text{blah2}(x)$ always False

translate quantified predicates

$\forall n \in \mathbb{Z}, n > 5$

For every integer n, n is greater than 5.

Every integer is greater than 5.

$\exists n \in \mathbb{Z}, n > 5$

There exists (at least one) integer n, where $n > 5$.

There exists (at least one) integer greater-than 5

There is an integer greater than 5

quantified binary predicates

$x + y = 17$: "sum of x, y is 17", where $x, y \in \mathbb{Z}$

$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 17$

$\forall x, y \in \mathbb{Z}, \ldots\ldots$

$\forall y, x \in \mathbb{Z}, x + y = 17$ — order doesn't matter here

also nothing prevents $x = y$

$\exists x, y \in \mathbb{Z}, x + y = 17$

True: ~~ex~~ $(x, y) = (9, 8)$

~~multiple~~ mixed — order matters quantifier examples

$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 17$

Let arbitrary $x \in \mathbb{Z}$ be fixed, choose $y = 17 - x$,

so $x + y = x + (17 - x) = 17$

$\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 17$

$y + 2 = 17$

$y + 3 = 17$

$y + 4 = 17$

~~order matters!~~ done!

$x + y = 17$

$\exists$: examples $\longrightarrow$ one example

$\forall$: lack of counterexamples –

negate quantified predicates

$\neg(\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x+y=17)$

same as $\forall y \in \mathbb{Z}, \neg(\forall x \in \mathbb{Z}, x+y=17)$

same as $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, \neg(x+y=17)$

use $\neq$ $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, x+y \neq 17$

given y, choose $x = 13{,}487{,}206 - y$

have predicates $P(x)$ and $Q(x)$

$\neg(\forall x \in S, P(x)) \equiv \exists x \in S, \neg P(x)$

$\neg(\exists x \in S, P(x)) \equiv \forall x \in S, \neg P(x)$

De Morgan $\neg(P(x) \wedge Q(x)) \equiv \neg P(x) \vee \neg Q(x)$

$\neg(P(x) \vee Q(x)) \equiv \neg P(x) \wedge \neg Q(x)$

manipulate negation

$$\neg(P(x) \Rightarrow Q(x)) \equiv P(x) \wedge \neg Q(x)$$

$$\equiv$$

$$\neg(\neg P(x) \vee Q(x)) \equiv \neg\neg P(x) \wedge \neg Q(x)$$

$$P(x) \wedge \neg Q(x)$$

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 2$$

$\neg(\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, x \geq 5 \vee x^2 - y \geq 30)$ False

$\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, x < 5 \wedge x^2 - y < 30$

Seems True, so

properties of integers, mostly $\mathbb{N}$ - Number Theory

$d, n \in \mathbb{N}$ what "d divides n" mean?

d divides n: "there is some $k \in \mathbb{Z}$, $n = kd$, write $d|n$", $d, n \in \mathbb{N}$

$3 \mid 15 \rightarrow$ True

$7 \mid 15 \rightarrow$ False because $15 = 2 \cdot 7 + 1$ and 1 is rem not 0.

divisibility

$\forall n \in \mathbb{N},\ n \mid 10 \Rightarrow n \mid 100$

True – if $\exists k \in \mathbb{Z},\ 10 = kn$, then $100 = \underbrace{10k}\, n$

$\forall n \in \mathbb{N}\ n \mid 100 \Rightarrow n \mid 10$

False $20 \mid 100$ but $10 = 20 \cdot 0 + \underline{\underline{10}}$

rem

primes

define $P(d)$: "$d > 1 \wedge \forall n \in \mathbb{N}, n | d \Rightarrow n \in \{d, 1\}$"

for $d \in \mathbb{N}$

- are there infinitely many primes
- $\forall n \in \mathbb{N}, P(n) \Rightarrow \exists y \in \mathbb{N}, P(y) \wedge y > n$
- Can you find a consecutive sequence of non-primes $n, n+1, n+2, \ldots, n+k$ for any $k \in \mathbb{N}$

Notes

$f(x) = 2x$ — agree onto

$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, 2x = y$

annotated week 1