Learning Objectives

By the end of this worksheet, you will:

- Translate sentences between natural English and predicate logic.

- Use mathematical definitions of predicates to simplify or expand formulas.

- Apply negation equivalence rules to simplify statements in predicate logic.

1. **Translation with predicates.** Suppose you have a set P of computer programs that are each meant to solve the same task. Some of the programs are written in Python, and some of the programs are written in a programming language that is not Python. Some of the programs correctly solve the task, and others do not.

   Let's define the following predicates:

   $$\text{Python}(x) : \text{``}x \text{ is a Python program''}, \text{ where } x \in P$$
   $$\text{Correct}(x) : \text{``}x \text{ solves the task correctly''}, \text{ where } x \in P$$

   Express each English statement below as a sentence in predicate logic. Note that all variables should be quantified over P (the set of computer programs that we're considering for this question).

   (a) Some incorrect program is written in Python.

   > **Solution**
   >
   > $\exists x \in P, \ \neg\text{Correct}(x) \wedge \text{Python}(x)$

   (b) Program my_prog is correct and is written in Python.

   > **Solution**
   >
   > $\text{Correct}(my\_prog) \wedge \text{Python}(my\_prog)$ There was some uncertainty about whether "Program my_prog..." was asserting membership of my_prog in P, so this would also be fine: $my\_prog \in P \wedge \text{Correct}(my\_prog) \wedge \text{Python}(my\_prog)$

   (c) No Python program is correct.

   > **Solution**
   >
   > $\neg\,(\exists x \in P, \ \text{Python}(x) \wedge \text{Correct}(x)) \quad$ or $\quad \forall x \in P, \ \text{Python}(x) \Rightarrow \neg\text{Correct}(x)$

   (d) Every incorrect program is written in Python.

   > **Solution**
   >
   > $\forall x \in P, \ \neg\text{Correct}(x) \Rightarrow \text{Python}(x) \quad$ or $\quad$ the contrapositive $\forall x \in P, \ \neg\text{Python}(x) \Rightarrow \text{Correct}(x)$

   Now, give a natural English sentence that captures the meaning of each symbolic sentence below.

   (a) $\exists x \in P, \ \text{Python}(x) \wedge \text{Correct}(x)$

   > **Solution**
   >
   > At least one Python program is correct.

   (b) $\forall x \in P, \ \neg\text{Python}(x) \wedge \text{Correct}(x)$

> **Solution**
>
> Every program is written in a language other than Python and is correct.

(c) $\neg\big(\forall x \in P,\ \mathrm{Correct}(x) \Rightarrow \mathrm{Python}(x)\big)$

> **Solution**
>
> Not every correct program is written in Python.
> *Alternate*: Some correct program is written in a language other than Python.

(d) $\forall x \in P,\ \neg\mathrm{Python}(x) \iff \mathrm{Correct}(x)$

> **Solution**
>
> For every program x, x is not written in Python if and only if x is correct.
> [Comment: this means the same thing as "For every program x, x is correct if and only if x is not written in Python."]

2. **Quantifiers in subformulas**. So far, we have generally seen quantifiers only as the leftmost components of our formulas. However, because all predicate statements have truth values (i.e., are either true or false), they too can be combined using the standard propositional operators. Let's see some examples of this.

   (a) Using the same predicates as Question 1, translate the following statement into English.

   $$\big(\forall x \in P,\ \mathrm{Python}(x) \Rightarrow \mathrm{Correct}(x)\big) \vee \big(\forall y \in P,\ \mathrm{Python}(y) \Rightarrow \neg\mathrm{Correct}(y)\big)$$

   > **Solution**
   >
   > All Python programs are correct, or all Python programs are incorrect.

   (b) Again using the same predicates as Question 1, translate the following statement into predicate logic. "If at least one Python program is correct, then all Python programs are correct."

   > **Solution**
   >
   > $\big(\exists x \in P,\ \mathrm{Python}(x) \wedge \mathrm{Correct}(x)\big) \Rightarrow \big(\forall y \in P,\ \mathrm{Python}(y) \Rightarrow \mathrm{Correct}(y)\big)$

   (c) Finally, consider the following two statements:

   $$\big(\exists x_1 \in \mathbb{N},\ x_1 \mid 165\big) \wedge \big(\exists x_2 \in \mathbb{N},\ 7 \mid x_2\big)$$

   $$\exists x \in \mathbb{N},\ x \mid 165 \wedge 7 \mid x$$

   What is the difference between these two statements? Are they true or false?

   > **Solution**
   >
   > In the first statement, there are two different numbers, and each one satisfies one predicate (dividing 165 or being divisible by 7). In the second statement, we only have one number, that must satisfy both predicates.
   > The first statement is true (why?) and the second statement is false (why?).

3. **Expanding definitions.** Consider the following statement:

   If m and n are odd integers, then mn is an odd integer.

If we want to express this statement using mathematical logic, we need to start with a definition of the term "odd". An integer n is said to be **odd** if and only if $2 \mid (n+1)$. That is, n is odd if and only if $\exists k \in \mathbb{Z}, n+1 = 2k$.

(a) Write the definition of a predicate over the integers named Odd that is True if and only if its argument is odd.

> **Solution**
>
> $$Odd(x) : \exists k \in \mathbb{Z}, x + 1 = 2k, \text{ where } x \in \mathbb{Z}.$$
>
> Or
>
> $$Odd(x) : 2 \mid (x+1), \text{where } x \in \mathbb{Z}.$$

(b) Using the predicate Odd and the notation of predicate logic, express the statement:

For every pair of odd integers m and n, mn is an odd integer.

> **Solution**
>
> $\forall m, n \in \mathbb{Z}, Odd(m) \wedge Odd(n) \Rightarrow Odd(mn)$

(c) Repeat part (b) but do not use the predicates Odd or $\mid$. Instead, use the full definition of Odd that includes a quantified statement.

> **Solution**
>
> $\forall m, n \in \mathbb{Z}, [(\exists k_1 \in \mathbb{Z}, m = 2k_1 + 1) \wedge (\exists k_2 \in \mathbb{Z}, n = 2k_2 + 1) \Rightarrow (\exists k_3 \in \mathbb{Z}, mn = 2k_3 + 1)]$

(d) Repeat parts (b) and (c) using the following statement (which states the converse of the original implication).

For every pair of integers m and n, if mn is odd, then m and n are odd.

> **Solution**
>
> $\forall m, n \in \mathbb{Z}, Odd(mn) \Rightarrow Odd(m) \wedge Odd(n)$
>
> Expanded definitions: $\forall m, n \in \mathbb{Z}, (\exists k_1 \in \mathbb{Z}, mn = 2k_1 + 1) \Rightarrow (\exists k_2 \in \mathbb{Z}, m = 2k_2 + 1) \wedge (\exists k_3 \in \mathbb{Z}, n = 2k_3 + 1)$

4. **Simplifying negated formulas.** Recall the rules governing how to simplify negations of predicate formulas:[1]

- $\neg(\neg p)$ becomes p.
- $\neg(p \vee q)$ becomes $(\neg p) \wedge (\neg q)$.
- $\neg(p \wedge q)$ becomes $(\neg p) \vee (\neg q)$.
- $\neg(p \Rightarrow q)$ becomes $p \wedge (\neg q)$.
- $\neg(p \Leftrightarrow q)$ becomes $(p \wedge (\neg q)) \vee ((\neg p) \wedge q)$.
- $\neg(\exists x \in S, \ P(x))$ becomes $\forall x \in S, \ \neg P(x)$.
- $\neg(\forall x \in S, \ P(x))$ becomes $\exists x \in S, \ \neg P(x)$.

Using these rules, simplify each of the following formulas so that the negations are applied directly to predicates/propositional variables. Note: this is a pretty mechanical exercise, but an extremely valuable one: once we get to the next chapter, we will be assuming you can take negations of statements very quickly as a first step in some proofs.

(a) $\neg\Big((a \wedge b) \Leftrightarrow c\Big)$

---

[1] Found on page 24 of the Course Notes.

> **Solution**
>
> $$\neg\Big((a \wedge b) \Leftrightarrow c\Big)$$
> $$\Big((a \wedge b) \wedge (\neg c)\Big) \vee \Big((\neg(a \wedge b) \wedge c)\Big)$$
> $$\Big(a \wedge b \wedge \neg c\Big) \vee \Big((\neg a \vee \neg b) \wedge c\Big)$$

(b) $\neg\Big(\forall x, y \in S, \ \exists z \in S, \ P(x,y) \wedge Q(x,z)\Big)$.

> **Solution**
>
> $$\neg\Big(\forall x, y \in S, \ \exists z \in S, \ P(x,y) \wedge Q(x,z)\Big)$$
> $$\exists x, y \in S, \ \forall z \in S, \ \neg\Big(P(x,y) \wedge Q(x,z)\Big)$$
> $$\exists x, y \in S, \ \forall z \in S, \ \neg P(x,y) \vee \neg Q(x,z)$$

(c) $\neg\Big((\exists x \in S, \ P(x)) \Rightarrow (\exists y \in S, \ Q(y))\Big)$.

> **Solution**
>
> $$\neg\Big((\exists x \in S, \ P(x)) \Rightarrow (\exists y \in S, \ Q(y))\Big)$$
> $$(\exists x \in S, \ P(x)) \wedge \neg(\exists y \in S, Q(y))$$
> $$(\exists x \in S, \ P(x)) \wedge (\forall y \in S, \neg Q(y))$$

5. **Choosing a universe and predicates.** Consider the statement

$$\Big[(\exists x \in U, P(x)) \wedge (\exists y \in U, Q(y))\Big] \Rightarrow \Big[\exists z \in U, P(z) \wedge Q(z)\Big].$$

Write down a non-empty domain U and predicates P and Q for which this statement is False.

Hint: Start by noting that the statement says: "If some $x \in U$ makes $P(x)$ True and some $y \in U$ makes $Q(y)$ True, then some $z \in U$ makes both $P(z)$ and $Q(z)$ True." Think about what must be so for the statement to be False, and use this to construct a U, P and Q.

> **Solution**
>
> Since we want to show that an implication is False, we want to construct an example where the hypothesis is True and the conclusion False.
>
> Let $U = \mathbb{Z}$ the set of integers. Define the predicates $P(x)$: 'x is even', where $x \in \mathbb{Z}$ and $Q(x)$: 'x is odd', where $x \in \mathbb{Z}$.
>
> Then $(\exists x \in U, P(x))$ is true, as the integer 2 provides an example. And $(\exists y \in U, \ Q(y))$ is also True, as the integer 3 provides an example. The hypothesis in the statement is therefore True, since the conjunction of two True values is True.
>
> However, there is no integer that is both even and odd, and so $[\exists \ z \in U, \ P(z) \wedge Q(z)]$ is false. (Put another way, the intersection of the even and odd numbers is empty.) The conclusion in the statement is False.
>
> For this example, we have an implication containing a True hypothesis and False conclusion. We can conclude

that the statement is False.