Learning objectives

By the end of this worksheet, you will:

- Know and apply various definitions for sets, strings, and common mathematical functions.
- Manipulate summation and product expressions.

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1. **Set complement.** Consider the two sets A and U and suppose $A \subseteq U$. The **complement of A in U**, denoted A^c, is the set of elements that are in U but not A. Notice that this depends on the choice of both U and A!

   (a) Let U be the set of natural numbers between 1 and 6, inclusive. Let $A = \{2, 5\}$. What is A^c?

   Solution

   $A^c = \{1, 3, 4, 6\}$.

   (b) Write an expression for A^c that uses the symbols A, U, and the set difference operator $\setminus$.

   Solution

   $A^c = U \setminus A$.

   (c) Let U represent the set of real numbers ($\mathbb{R}$), and consider the sets $A = \{x \mid x \in U \text{ and } 0 < x \leq 2\}$ and $B = \{x \mid x \in U \text{ and } 1 \leq x < 4\}$. Find each of the following, where the complement is taken with respect to U: $A^c \cap B^c$, $A^c \cup B^c$, $(A \cap B)^c$ and $(A \cup B)^c$. Drawing number lines may be helpful. Any observations?

   Solution

   $A^c \cap B^c = \{x \mid x \in U \text{ and } x \leq 0 \text{ or } x \geq 4\}$,
   $A^c \cup B^c = \{x \mid x \in U \text{ and } x < 1 \text{ or } x > 2\}$,
   $(A \cap B)^c = \{x \mid x \in U \text{ and } x < 1 \text{ or } x > 2\}$,
   $(A \cup B)^c = \{x \mid x \in U \text{ and } x \leq 0 \text{ or } x \geq 4\}$.
   Note that : $A^c \cap B^c = (A \cup B)^c$ and $A^c \cup B^c = (A \cap B)^c$ (de Morgan's laws for sets)

2. **Set partitions.** A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is called a **partition** of a set A if and only if (1) A is the union of all of the A_i,[1] and (2) the sets $A_1, A_2, A_3, \ldots$ do not have any elements in common.[2]

   (a) Let $\mathbb{Z}^+$ be the set of all positive integers, and let

   $$\begin{aligned} T_0 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k, \text{ for some integer } k\}, \\ T_1 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k+1, \text{ for some integer } k\}, \\ T_2 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k+2, \text{ for some integer } k\}, \\ T_3 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 6k, \text{ for some integer } k\}. \end{aligned}$$

   Write the first three elements of T_0, of T_1, of T_2, and of T_3.

   Solution

   $T_0 = \{3, 6, 9, \ldots\}$, $T_1 = \{1, 4, 7, \ldots\}$, $T_2 = \{2, 5, 8, \ldots\}$, $T_3 = \{6, 12, 18, \ldots\}$.

   (b) Write down a partition of $\mathbb{Z}^+$ using T_0, T_1, T_2, and/or T_3. Why can't you use all four sets?

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[1] We say the A_i are exhaustive.

[2] We say the A_i are **mutually disjoint (or pairwise disjoint or nonoverlapping)** if and only if no two sets A_i and A_j with distinct subscripts have any elements in common.

Solution

The set $\{T_0, T_1, T_2\}$ is a partition of $\mathbb{Z}^+$, since, when any positive integer is divided by 3, the possible integer remainders are 0, 1, and 2. The sets T_0, T_1, T_2 list the numbers whose remainder when divided by 3 are 0, 1, or 2, respectively.

Note that $T_3 \subseteq T_0$, so we can't use both T_0 and T_3 in our partition (they have elements in common).

3. **Strings.** An **alphabet** A is a set of symbols like $\{0, 1\}$ or $\{a, b, c\}$. A **string over alphabet** A is a finite sequence of elements from A; the *length* of a string is simply the number of elements. Order matters in a string.

   For example, 011 is a string over $\{0, 1\}$ of length three, and $abbbacc$ is a string over $\{a, b, c\}$ of length seven.

   (a) Write down all strings over the alphabet $\{0, 1\}$ of length three (you should have eight in total).

   Solution

   $\{000, 001, 010, 011, 100, 101, 110, 111\}$

   (b) Let S_1 be the set of all strings over $\{a, b, c\}$ that have length two, and S_2 be the set of all strings over $\{a, b, c\}$ that start and end with the same letter. Find $S_1 \cap S_2$ and $S_1 \backslash S_2$.

   Solution

   $S_1 \cap S_2 = \{aa, bb, cc\}$.
   $S_1 \backslash S_2 = \{ab, ac, ba, bc, ca, cb\}$.

   (c) What do you notice about the relationship between S_1, $S_1 \cap S_2$, and $S_1 \backslash S_2$?

   Solution

   Hint: look at $(S_1 \cap S_2) \cup (S_1 \backslash S_2)$.

4. **The floor and ceiling functions.** Given any real number x, the **floor of** x, denoted $\lfloor x \rfloor$, is defined to be the largest integer that is less than or equal to x. Similarly, the **ceiling of** x, denoted $\lceil x \rceil$, is defined to be the smallest integer that is greater than or equal to x.

   (a) What is the domain and range of the floor and ceiling functions?

   Solution

   The domain is $\mathbb{R}$ and the range is $\mathbb{Z}$.

   (b) Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of x: $x = \frac{25}{4}$, $x = 0.999$, and $x = -2.01$.

   Solution

   $\left\lfloor \frac{25}{4} \right\rfloor = \lfloor 6.25 \rfloor = 6$, $\left\lceil \frac{25}{4} \right\rceil = \lceil 6.25 \rceil = 7$, $\lfloor 0.999 \rfloor = 0$, $\lceil 0.999 \rceil = 1$, $\lfloor -2.01 \rfloor = -3$, $\lceil -2.01 \rceil = -2$.

   (c) Consider the following statement: For all real numbers x and y, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$. Do you think this statement is True or False? Why?

   Solution

   The statement is False, since, for example, $\left\lfloor \frac{1}{2} + \frac{2}{3} \right\rfloor = \left\lfloor \frac{7}{6} \right\rfloor = 1$, while $\left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor = 0 + 0 = 0$.

5. Recall that the notation $\sum_{i=j}^{k} f(i)$ gives us a short form for expressing the sum $f(j) + f(j+1) + \cdots + f(k-1) + f(k)$, and that $\prod_{i=j}^{k} f(i)$ gives us a short form for expressing the product $f(j) \times f(j+1) \times \cdots \times f(k-1) \times f(k)$.

   (a) Expand the following expressions to get the long sum/product they represent. Do not simplify.

Solution

$$\sum_{k=1}^{3}(k+1) = (1+1) + (2+1) + (3+1)$$

$$\sum_{k=-1}^{2}(k^2+3) = (1+3) + (0+3) + (1+3) + (4+3)$$

$$\sum_{k=1}^{5}(2k) = 2 + 4 + 6 + 8 + 10$$

$$\sum_{m=0}^{1} \frac{1}{2^m} = \frac{1}{2^0} + \frac{1}{2^1}$$

$$\sum_{j=0}^{4}(-1)^j \frac{j}{j+1} = 0 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5}$$

$$\prod_{i=2}^{4} \frac{i(i+2)}{(i-1)(i+1)} = \frac{2\cdot 4}{1\cdot 3} \cdot \frac{3\cdot 5}{2\cdot 4} \cdot \frac{4\cdot 6}{3\cdot 5}$$

   (b) Simplify each of the following expressions by using $\sum$ or $\prod$ notation.

Solution

$$3 + 6 + 12 + 24 + 48 + 96 = \sum_{i=0}^{5} 3 \cdot 2^i$$

$$\frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \frac{25}{243} + \frac{36}{729} = \sum_{j=1}^{6} \frac{j^2}{3^j}$$

$$0 + 1 - 2 + 3 - 4 + 5 = \sum_{j=0}^{5}(-1)^{j+1} \cdot j$$

$$\left(\frac{1}{1+1}\right) \times \left(\frac{2}{2+1}\right) \times \left(\frac{3}{3+1}\right) \times \cdots \times \left(\frac{k}{k+1}\right) = \prod_{j=1}^{k}\left(\frac{j}{j+1}\right)$$

$$\left(\frac{1\cdot 2}{3\cdot 4}\right) \times \left(\frac{2\cdot 3}{4\cdot 5}\right) \times \left(\frac{3\cdot 4}{5\cdot 6}\right)$$
$$= \prod_{j=1}^{3} \frac{j\cdot(j+1)}{(j+2)\cdot(j+3)}$$

6. It is not to hard to prove manipulation results like the following that can be used to help us manipulate sums and products. If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

$$\sum_{k=m}^{n}(a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

$$\sum_{k=m}^{n} c \cdot a_k = c \cdot \sum_{k=m}^{n} a_k$$

$$\prod_{k=m}^{n}(a_k \cdot b_k) = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right)$$

Using these laws, rewrite each of the following as a single sum or product, but do not simplify your final sum/product. (You'll learn late in the course how to do so.)

Solution

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\begin{aligned}
3 \cdot \sum_{k=1}^{n}(2k-3) + \sum_{k=1}^{n}(4-5k) &= 6 \cdot \left(\sum_{k=1}^{n} k\right) - 9 \cdot \left(\sum_{k=1}^{n} 1\right) + 4 \cdot \left(\sum_{k=1}^{n} 1\right) - 5 \cdot \left(\sum_{k=1}^{n} k\right) \\
&= \sum_{k=1}^{n}(k-5)
\end{aligned}
$$

$$
\begin{aligned}
\left(\prod_{k=1}^{n} \frac{k}{k+1}\right)\left(\prod_{k=1}^{n} \frac{k+1}{k+2}\right) &= \left(\prod_{k=1}^{n} \frac{k}{k+1} \cdot \frac{k+1}{k+2}\right) \\
&= \left(\prod_{k=1}^{n} \frac{k}{k+2}\right)
\end{aligned}
$$