Learning objectives

By the end of this worksheet, you will:

- Know and apply various definitions for sets, strings, and common mathematical functions.
- Manipulate summation and product expressions.

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1. **Set complement**. Consider the two sets A and U and suppose $A \subseteq U$. The **complement of** A **in** U, denoted A^c, is the set of elements that are in U but not A. Notice that this depends on the choice of **both** U and A!

   (a) Let U be the set of natural numbers between 1 and 6, inclusive. Let $A = \{2, 5\}$. What is A^c?

   (b) Write an expression for A^c that uses the symbols A, U, and the set difference operator $\setminus$.

   (c) Let U represent the set of real numbers ($\mathbb{R}$), and consider the sets $A = \{x \mid x \in U \text{ and } 0 < x \leq 2\}$ and $B = \{x \mid x \in U \text{ and } 1 \leq x < 4\}$. Find each of the following, where the complement is taken with respect to U: $A^c \cap B^c$, $A^c \cup B^c$, $(A \cap B)^c$ and $(A \cup B)^c$. Drawing number lines may be helpful. Any observations?

2. **Set partitions**. A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is called a **partition** of a set A if and only if (1) A is the union of all of the A_i,[1] and (2) the sets $A_1, A_2, A_3, \ldots$ do not have any elements in common.[2]

   (a) Let $\mathbb{Z}^+$ be the set of all positive integers, and let

$$\begin{aligned} T_0 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k, \text{ for some integer } k\}, \\ T_1 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k+1, \text{ for some integer } k\}, \\ T_2 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k+2, \text{ for some integer } k\}, \\ T_3 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 6k, \text{ for some integer } k\}. \end{aligned}$$

   Write the first three elements of T_0, of T_1, of T_2, and of T_3.

   (b) Write down a partition of $\mathbb{Z}^+$ using T_0, T_1, T_2, and/or T_3. Why can't you use all four sets?

[1] We say the A_i are exhaustive.

[2] We say the A_i are **mutually disjoint (or pairwise disjoint or nonoverlapping)** if and only if no two sets A_i and A_j with distinct subscripts have any elements in common.

3. **Strings.** An **alphabet** A is a set of symbols like $\{0, 1\}$ or $\{a, b, c\}$. A **string over alphabet** A is a finite sequence of elements from A; the *length* of a string is simply the number of elements. Order matters in a string.

   For example, 011 is a string over $\{0, 1\}$ of length three, and $abbbacc$ is a string over $\{a, b, c\}$ of length seven.

   (a) Write down all strings over the alphabet $\{0, 1\}$ of length three (you should have eight in total).

   (b) Let S_1 be the set of all strings over $\{a, b, c\}$ that have length two, and S_2 be the set of all strings over $\{a, b, c\}$ that start and end with the same letter. Find $S_1 \cap S_2$ and $S_1 \backslash S_2$.

   (c) What do you notice about the relationship between S_1, $S_1 \cap S_2$, and $S_1 \backslash S_2$?

4. **The floor and ceiling functions.** Given any real number x, the **floor of** x, denoted $\lfloor x \rfloor$, is defined to be the largest integer that is less than or equal to x. Similarly, the **ceiling of** x, denoted $\lceil x \rceil$, is defined to be the smallest integer that is greater than or equal to x.

   (a) What is the domain and range of the floor and ceiling functions?

   (b) Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of x: $x = \frac{25}{4}$, $x = 0.999$, and $x = -2.01$.

   (c) Consider the following statement: For all real numbers x and y, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$. Do you think this statement is True or False? Why?

5. Recall that the notation $\sum_{i=j}^{k} f(i)$ gives us a short form for expressing the sum $f(j)+f(j+1)+\cdots+f(k-1)+f(k)$, and that $\prod_{i=j}^{k} f(i)$ gives us a short form for expressing the product $f(j)\times f(j+1)\times\cdots\times f(k-1)\times f(k)$.

   (a) Expand the following expressions to get the long sum/product they represent. Do not simplify.

   $\sum_{k=1}^{3}(k+1)$ $\qquad$ $\sum_{m=0}^{1}\frac{1}{2^m}$

   $\sum_{k=-1}^{2}(k^2+3)$ $\qquad$ $\sum_{j=0}^{4}(-1)^j\frac{j}{j+1}$

   $\sum_{k=1}^{5}(2k)$ $\qquad$ $\prod_{i=2}^{4}\frac{i(i+2)}{(i-1)(i+1)}$

   (b) Simplify each of the following expressions by using $\sum$ or $\prod$ notation.

   $3+6+12+24+48+96$ $\qquad$ $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}+\frac{25}{243}+\frac{36}{729}$

   $0+1-2+3-4+5$ $\qquad$ $\left(\frac{1}{1+1}\right)\times\left(\frac{2}{2+1}\right)\times\left(\frac{3}{3+1}\right)\times\cdots\times\left(\frac{k}{k+1}\right)$

   $\left(\frac{1\cdot 2}{3\cdot 4}\right)\times\left(\frac{2\cdot 3}{4\cdot 5}\right)\times\left(\frac{3\cdot 4}{5\cdot 6}\right)$

6. It is not to hard to prove manipulation results like the following that can be used to help us manipulate sums and products. If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

$$\sum_{k=m}^{n}(a_k+b_k)=\sum_{k=m}^{n}a_k+\sum_{k=m}^{n}b_k$$

$$\sum_{k=m}^{n}c\cdot a_k=c\cdot\sum_{k=m}^{n}a_k$$

$$\prod_{k=m}^{n}(a_k\cdot b_k)=\left(\prod_{k=m}^{n}a_k\right)\left(\prod_{k=m}^{n}b_k\right)$$

Using these laws, rewrite each of the following as a single sum or product, but do not simplify your final sum/product. (You'll learn late in the course how to do so.)

$3\cdot\sum_{k=1}^{n}(2k-3)+\sum_{k=1}^{n}(4-5k)$ $\qquad$ $\left(\prod_{k=1}^{n}\frac{k}{k+1}\right)\left(\prod_{k=1}^{n}\frac{k+1}{k+2}\right)$