SAMPLE SOLUTIONS

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UNIVERSITY OF TORONTO
Faculty of Arts and Science

Term test #2

CSC 165H1
Duration — 50 minutes

aids allowed: 8.5" x 11" handwritten aid sheet, both sides

Last Name: ____________________

First Name: ____________________

Do **not** *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 3 questions on 5 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question," on.

Good Luck!

Question 1. [10 MARKS]

Use the proof structure from this course (including comments) to **prove**:

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, (\exists i \in \mathbb{N}, m = 7i + 3) \wedge (\exists j \in \mathbb{N}, n = 7j + 2) \Rightarrow (\exists k \in \mathbb{N}, m + n = 7k + 5)$$

Proof:

Assume $m \in \mathbb{N}, n \in \mathbb{N}$ # generic natural numbers

Assume $(\exists i \in \mathbb{N}, m = 7i + 3) \wedge (\exists j \in \mathbb{N}, n = 7j + 2)$ # the antecedent

Let $i_0 \in \mathbb{N}$ be such that $m = 7i_0 + 3$

Let $j_0 \in \mathbb{N}$ be such that $n = 7j_0 + 2$

then $m + n = (7i_0 + 3) + (7j_0 + 2) = 7(i_0 + j_0) + 5$

Let $k = i_0 + j_0$

then $k \in \mathbb{N}$ # i_0 and j_0 are both natural numbers

then $m + n = 7k + 5$

then $\exists k \in \mathbb{N}, m + n = 7k + 5$

then $(\exists i \in \mathbb{N}, m = 7i + 3) \wedge (\exists j \in \mathbb{N}, n = 7j + 2) \Rightarrow \exists k \in \mathbb{N}, m + n = 7k + 5$

then $\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, (\exists i \in \mathbb{N}, m = 7i + 3) \wedge (\exists j \in \mathbb{N}, n = 7j + 2) \Rightarrow (\exists k \in \mathbb{N}, m + n = 7k + 5)$

Question 2. [10 MARKS]

For $x \in \mathbb{R}$, define $\lfloor x \rfloor$ by

$$\lfloor x \rfloor \in \mathbb{Z} \wedge \lfloor x \rfloor \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq \lfloor x \rfloor)$$

Use this definition of $\lfloor x \rfloor$, and the structured proof technique from this course (including comments, where appropriate), to **disprove**:

$$\exists e \in \mathbb{R}^+, \forall d \in \mathbb{R}^+, \exists x \in \mathbb{R}, (x > d) \wedge (\lfloor x \rfloor \leq e)$$

Sample solution: First negate the statement, which yields the following

$$\forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > d \Rightarrow \lfloor x \rfloor > e$$

Now we prove this negation.

Proof:

Assume $e \in \mathbb{R}^+$ # generic positive real number

Pick $d = e + 1$, then $d \in \mathbb{R}^+$

Assume $x \in \mathbb{R}$ # generic real number

Assume $x > d$ # the antecedent

then $x > e + 1$ # because of the d we picked

Assume $\lfloor x \rfloor \leq e$ # for contradiction

then $\lfloor x \rfloor + 1 \leq e + 1$ # add 1 to both side

then $\lfloor x \rfloor + 1 < x$ # transitivity

and $\lfloor x \rfloor + 1 \in \mathbb{Z}$ # $\lfloor x \rfloor$ and 1 are both integers

then $\lfloor x \rfloor + 1 \leq \lfloor x \rfloor$ # by definition of $\lfloor x \rfloor$

then $1 \leq 0$ # subtract $\lfloor x \rfloor$ from both sides, and contradict with $1 > 0$

then $\lfloor x \rfloor > e$ # because of contradiction

then $x > d \Rightarrow \lfloor x \rfloor > e$ # introduce $\Rightarrow$

then $\forall x \in \mathbb{R}, x > d \Rightarrow \lfloor x \rfloor > e$ # introduce $\forall$

then $\exists d \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > d \Rightarrow \lfloor x \rfloor > e$ # introduce $\exists$

then $\forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > d \Rightarrow \lfloor x \rfloor > e$ # introduce $\forall$

Alternative Proof utilising the result $\lfloor x \rfloor > x - 1$

Assume $e \in \mathbb{R}^+$ # generic positive real number

Pick $d = e + 1$, then $d \in \mathbb{R}^+$

Assume $x \in \mathbb{R}$ # generic real number

Assume $x > d$ # the antecedent

then $x > e + 1$ # because of the d we picked

then $x - 1 > e$ # subtract 1 from both sides

and $\lfloor x \rfloor > x - 1$ # property proven in lecture

then $\lfloor x \rfloor > e$ # transitivity

$$\text{then } x > d \Rightarrow \lfloor x \rfloor > e \text{ \# introduce } \Rightarrow$$
$$\text{then } \forall x \in \mathbb{R}, x > d \Rightarrow \lfloor x \rfloor > e \text{ \# introduce } \forall$$
$$\text{then } \exists d \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > d \Rightarrow \lfloor x \rfloor > e \text{ \# introduce } \exists$$
$$\text{then } \forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > d \Rightarrow \lfloor x \rfloor > e \text{ \# introduce } \forall$$

Question 3. [10 MARKS]

For $x \in \mathbb{R}$, define $\lfloor x \rfloor$ by

$$\lfloor x \rfloor \in \mathbb{Z} \wedge \lfloor x \rfloor \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq \lfloor x \rfloor)$$

Use this definition of $\lfloor x \rfloor$, and the proof structure from this course (including comments) to **prove**:

$$\forall z \in \mathbb{Z}, \forall x \in \mathbb{R}, \lfloor x \rfloor + z = \lfloor x + z \rfloor$$

Sample solution: Like what we did in the assignment, we prove this equality by showing

$$(\lfloor x \rfloor + z \leq \lfloor x + z \rfloor) \wedge (\lfloor x + z \rfloor \leq \lfloor x \rfloor + z)$$

Proof:

Assume $z \in \mathbb{Z}$ # generic integer

  Assume $x \in \mathbb{R}$ # generic real number

    then $\lfloor x \rfloor \leq x$ # by definition of $\lfloor x \rfloor$

    then $\lfloor x \rfloor + z \leq x + z$ # add x to both sides

    and $\lfloor x \rfloor + z \in \mathbb{Z}$ # both $\lfloor x \rfloor$ and z are integers

    then $\lfloor x \rfloor + z \leq \lfloor x + z \rfloor$ # definition of $\lfloor x + z \rfloor$, first inequality obtained

    and $\lfloor x + z \rfloor \leq x + z$ # definition of $\lfloor x + z \rfloor$

    then $\lfloor x + z \rfloor - z \leq x$ # subtract z from both sides

    and $\lfloor x + z \rfloor - z \in \mathbb{Z}$ # both $\lfloor x + z \rfloor$ and z are integers

    then $\lfloor x + z \rfloor - z \leq \lfloor x \rfloor$ # definition of $\lfloor x \rfloor$

    then $\lfloor x + z \rfloor \leq \lfloor x \rfloor + z$ # add z to both sides, second inequality obtained

    then $(\lfloor x \rfloor + z \leq \lfloor x + z \rfloor) \wedge (\lfloor x + z \rfloor \leq \lfloor x \rfloor + z)$ # conjunction introduction

    then $\lfloor x \rfloor + z = \lfloor x + z \rfloor$ # $a = b \Leftrightarrow a \leq b \wedge b \leq a$

  then $\forall x \in \mathbb{R}, \lfloor x \rfloor + z = \lfloor x + z \rfloor$ # introduce $\forall$

then $\forall z \in \mathbb{Z}, \forall x \in \mathbb{R}, \lfloor x \rfloor + z = \lfloor x + z \rfloor$ # introduce $\forall$

1: _______/10

2: _______/10

3: _______/10

TOTAL: _______/30