QUESTION 1. [7 MARKS]

Read the following statement:

$$S1: \quad \forall m \in \mathbb{N}, (\exists q_1 \in \mathbb{N}, m = 6q_1 + 2) \Rightarrow \left(\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4\right)$$

$S1$ is true. Use the proof structure from this course to PROVE it.

SAMPLE SOLUTION:

Assume $m \in \mathbb{N}$. # in order to introduce $\forall$

Assume $\exists q_1 \in \mathbb{N}, m = 6q_1 + 2$. # in order to introduce $\Rightarrow$

Then $m^2 = (6q_1 + 2)^2 = 36q_1^2 + 24q_1 + 4$ # substitute $m = 6q_1 + 2$ into m^2.
Then $m^2 = 6(6q_1^2 + 4q_1) + 4$. # factor previous line
Then $\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4$.
Pick $q_2 = 6q_1^2 + 4q_1 \in \mathbb{N}$, since $6, q_1, 4 \in \mathbb{N}$, which is closed under $+, \times$.
introduced $\exists$

Then $\exists q_1 \in \mathbb{N}, m = 6q_1 + 2 \Rightarrow \left(\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4\right)$ # introduced $\Rightarrow$

Then $\forall m \in \mathbb{N}, (\exists q_1 \in \mathbb{N}, m = 6q_1 + 2) \Rightarrow \left(\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4\right)$. # introduced $\forall$

QUESTION 2. [6 MARKS]

Read the following statement:

$$S2: \quad \forall m \in \mathbb{N}, \left(\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4\right) \Rightarrow (\exists q_1 \in \mathbb{N}, m = 6q_1 + 2)$$

$S2$ is false. Use the proof structure from this course to DISPROVE it.

SAMPLE SOLUTION: I negate $S2$ and then prove the negation:

$$\neg S2: \quad \exists m \in \mathbb{N}, \left(\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4\right) \wedge \forall q_1 \in \mathbb{N}, m \neq 6q_1 + 2$$

Pick $m = 4$. Then $m \in \mathbb{N}$ # $6 \in \mathbb{N}$.

Also $4^2 = 16 = 6 \times 2 + 4$.

Then $\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4$. # Just pick $q_2 = 2$

Also $m = 6 \times 0 + 4$, so $(0, 4)$ is the unique (q, r) pair satisfying $m = 6q + r$ and $6 > r \geq 0$.

by definition of quotient/remainder in mathematical prerequisites

Then $\forall q_1 \in \mathbb{N}, m \neq 6q_1 + 2$. # since $(0, 4)$ is unique.

Then $\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4 \wedge \forall q_1 \in \mathbb{N}, m \neq 6q_1 + 2$

introduced $\exists$

Question 3. [6 marks]

Read the definition of $\lfloor x \rfloor$ below:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor \in \mathbb{Z} \wedge \lfloor x \rfloor \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq \lfloor x \rfloor)$$

Use the proof structure from this course, and the definition of $\lfloor x \rfloor$ above, to prove the following statement:

$$S3: \quad \forall x \in \mathbb{R}, \lfloor x \rfloor + 7 > x + 6$$

Sample solution:

Assume x is a generic real number # in order to introduce $\forall$

Then $\lfloor x \rfloor \in \mathbb{Z}$. # definition of $\lfloor x \rfloor$
Then $\lfloor x \rfloor + 1 \in \mathbb{Z}$. # $\lfloor x \rfloor, 1 \in \mathbb{Z}$, and $\mathbb{Z}$ closed under $+$
Then $\lfloor x \rfloor + 1 > \lfloor x \rfloor$. # add $\lfloor x \rfloor$ to $1 > 0$
Then $\lfloor x \rfloor + 1 > x$. # by contrapositive of $(\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq \lfloor x \rfloor)$
Then $\lfloor x \rfloor + 7 > x + 6$. # add 6 to previous line

Conclude $\forall x \in \mathbb{R}, \lfloor x \rfloor + 7 > x + 6$. # introduced $\forall$

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1: ______/ 7

2: ______/ 6

3: ______/ 6

TOTAL: ______/19