1. Write a detailed, structured proof that

$$\forall f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, \forall g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, g \in \Omega(f) \Rightarrow g^2 \in \Omega(f^2)$$

(where f^2 and g^2 are defined in the obvious way: $\forall n \in \mathbb{N}, f^2(n) = f(n) \cdot f(n)$, and similarly for g).

(I show only the finished proof here, not its development.)

Assume $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$ and $g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$.

    Assume $g \in \Omega(f)$.

        Then $\exists c_0 \in \mathbb{R}^+, \exists B_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B_0 \Rightarrow g(n) \geqslant c_0 \cdot f(n)$. # definition of Ω

        # *Show that* $g^2 \in \Omega(f^2)$:

        Let $c_1 = c_0^2$. Then $c_1 \in \mathbb{R}^+$. # because $c_0 \in \mathbb{R}^+$

        Let $B_1 = B_0$. Then $B_1 \in \mathbb{N}$. # because $B_0 \in \mathbb{N}$

        Assume $n \in \mathbb{N}$ and $n \geqslant B_1 = B_0$.

            Then $g(n) \geqslant c_0 \cdot f(n)$ (because $n \geqslant B_0$),

            so $g^2(n) = g(n) \cdot g(n) \geqslant (c_0 \cdot f(n)) \cdot (c_0 \cdot f(n)) = c_0^2 \cdot f(n) \cdot f(n) = c_1 \cdot f^2(n)$.

        Hence, $\forall n \in \mathbb{N}, n \geqslant B_1 \Rightarrow g^2(n) \geqslant c_1 \cdot f^2(n)$.

        Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g^2(n) \geqslant c \cdot f^2(n)$.

        Thus, $g^2 \in \Omega(f^2)$. # by definition of Ω

    Therefore, $g \in \Omega(f) \Rightarrow g^2 \in \Omega(f^2)$.

Then, $\forall f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, \forall g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, g \in \Omega(f) \Rightarrow g^2 \in \Omega(f^2)$.

2. Prove or disprove the following statement:

$$\forall f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, \forall g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, f \in \mathcal{O}(g) \Rightarrow (f+g) \in \Theta(g)$$

(where $(f+g)$ is defined in the obvious way: $\forall n \in \mathbb{N}, (f+g)(n) = f(n) + g(n)$).

(I show only the finished proof here, not its development. Another way to write this proof is to prove separately $(f+g) \in \Omega(g)$ and $(f+g) \in \mathcal{O}(g)$, using c_1 and c_2, respectively.)

Assume $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$ and $g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$.
  Assume $f \in \mathcal{O}(g)$.
    Then $\exists c_0 \in \mathbb{R}^+, \exists B_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B_0 \Rightarrow f(n) \leqslant c_0 \cdot g(n)$. # definition of $\mathcal{O}$
    # *Show that* $(f+g) \in \Theta(g)$:
    Let $c_1 = 1$ and $c_2 = c_0 + 1$. Then $c_1 \in \mathbb{R}^+$ and $c_2 \in \mathbb{R}^+$. # because $c_0 \in \mathbb{R}^+$
    Let $B_1 = B_0$. Then $B_1 \in \mathbb{N}$. # because $B_0 \in \mathbb{N}$
    Assume $n \in \mathbb{N}$ and $n \geqslant B_1$.
      Then $c_1 g(n) = g(n) \leqslant f(n) + g(n) = (f+g)(n)$. # because $f(n) \geqslant 0$
      Also, $(f+g)(n) = f(n) + g(n) \leqslant c_0 g(n) + g(n) = c_2 g(n)$. # because $n \geqslant B_1 = B_0$
    Hence, $\forall n \in \mathbb{N}, n \geqslant B_1 \Rightarrow c_1 g(n) \leqslant (f+g)(n) \leqslant c_2 g(n)$.
    Then $\exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B_1 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B_1 \Rightarrow c_1 g(n) \leqslant (f+g)(n) \leqslant c_2 g(n)$.
    So $(f+g) \in \Theta(g)$. # by definition
  Then $f \in \mathcal{O}(g) \Rightarrow (f+g) \in \Theta(g)$.
Then $\forall f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, \forall g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}, f \in \mathcal{O}(g) \Rightarrow (f+g) \in \Theta(g)$.