1. Write the *complete proof* of each of the following statements. Note that, for the first two statements, we have already written down the proof structures in previous tutorial.

   (a) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \leqslant y \Rightarrow \exists z \in \mathbb{Z}, x \leqslant z \leqslant y$

   Assume $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$.
       Assume $x \leqslant y$.
           Let $z' = y$.
           Then, $z' \in \mathbb{Z}$. # because $y \in \mathbb{Z}$
           Then, $x \leqslant z'$ and $z' \leqslant y$. # because $x \leqslant y$
           Then, $\exists z \in \mathbb{Z}, x \leqslant z \leqslant y$.
       Then, $x \leqslant y \Rightarrow \exists z \in \mathbb{Z}, x \leqslant z \leqslant y$.
   Then, $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \leqslant y \Rightarrow \exists z \in \mathbb{Z}, x \leqslant z \leqslant y$.

   (b) $\forall x \in \mathbb{Z}, (\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)$

   Assume $x \in \mathbb{Z}$.
       Assume $\exists y \in \mathbb{Z}, x = 3y + 1$.
           Let y_0 be such that $x = 3y_0 + 1$. # by assumption
           Let $y' = 3y_0^2 + 2y_0$.
           Then, $y' \in \mathbb{Z}$.
           Then, $x^2 = (3y_0+1)^2 = (3y_0)^2+2(1)(3y_0)+1^2 = 9y_0^2+6y_0+1 = 3(3y_0^2+2y_0)+1 = 3y'+1$.
           Then, $\exists y \in \mathbb{Z}, x^2 = 3y + 1$.
       Then, $(\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)$.
   Then, $\forall x \in \mathbb{Z}, (\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)$.

   (c) $\neg(\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, y > x \wedge a_y > a_x)$ — for the sequence $A = 2, 4, 6, 8, 9, 7, 5, 3, 1, 0, 0, 0, 0, 0, \ldots$

   This statement is equivalent to:

   $$\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, y > x \Rightarrow a_y \leqslant a_x$$

   Let $x' = 8$.
   Then, $x' \in \mathbb{N}$.
   Assume $y \in \mathbb{N}$.
       Assume $y > x' = 8$.
           Then, $a_y = 0$. # because $a_9 = a_{10} = a_{11} = \cdots = 0$
           Then, $a_y \leqslant 1 = a_8 = a_{x'}$.
       Then, $y > x' \Rightarrow a_y \leqslant a_{x'}$.
   Then, $\forall y \in \mathbb{N}, y > x' \Rightarrow a_y \leqslant a_{x'}$.
   Then, $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, y > x \Rightarrow a_y \leqslant a_x$.