office hour for test: tomorrow 3-5 in BA3201

Term test:
next Wed or
earlier

CSC165 fall 2014

Mathematical expression

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Course notes, chapter 4

Outline

Maximum slice sum

polynomials

notes

annotated slides

maximum slice

$n = len(L)$

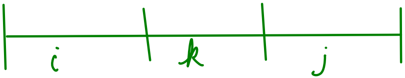
```
def max_sum(L) :
  """maximum sum over slices of L"""
  max = 0
  i = 0
  while i < len(L) :
    j = i + 1
    while j <= len(L) :
      sum = 0
      k = i
      while k < j :
        sum = sum + L[k]
        k = k + 1
      if sum > max :
        max = sum
      j = j + 1
    i = i + 1
  return max
```

- no more than 1 step
- no more than 1 step for each value of i, or n
- no more than n steps for each i, or n^2 steps
- no more than n steps for each i, or n^3 steps altogether.

- "loop guard"
- no more than n loop guards
- no more than n^2 loop guards.

maximum slice

Want to show $W_{MS}(n) \le c \cdot n^3$

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      if sum > max :
        max = sum
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    i = i + 1
  return max
```

restrict consideration of i to $i = 0, \ldots, \lceil n/3 \rceil - 1 \rightarrow \lceil n/3 \rceil$ values.

restrict j's to:

$n - \lceil n/3 \rceil + 1, \ldots, n \rightarrow \lceil n/3 \rceil$ values

restrict k to

$\lceil n/3 \rceil - 1, \ldots, n - \lceil n/3 \rceil$

$\rightarrow n + 1 - \lceil n/3 \rceil - \lceil n/3 \rceil + 1$

$= n + 2 - 2\lceil n/3 \rceil \ge \lceil n/3 \rceil$

$\Longleftrightarrow n + 2 \ge 3\lceil n/3 \rceil$

$* \quad \lceil \frac{n}{3} \rceil \le \frac{n+2}{3}$

So... $n + 2 = \frac{3n+6}{3} \ge 3\left(\frac{n+2}{3}\right) \ge 3\lceil n/3 \rceil$

$\forall x \in \mathbb{R}, \quad \lceil x \rceil \in \mathbb{Z} \wedge \lceil x \rceil \ge x \wedge (\forall z \in \mathbb{Z}, z \ge x \Rightarrow z \ge \lceil x \rceil)$

maximum slice

```
def max_sum(L) :
  """maximum sum over slices of L"""
  max = 0
  i = 0
  while i < len(L) :
    j = i + 1
    while j <= len(L) :
      sum = 0
      k = i
      while k < j :
        sum = sum + L[k]
        k = k + 1
      if sum > max :
        max = sum
      j = j + 1
    i = i + 1
  return max
```

So, we know there are at least $\lceil n/3 \rceil \times \lceil n/3 \rceil \times \lceil n/3 \rceil$

is, ks, js "visited"

for $\frac{n}{3} \times \frac{n}{3} \times \frac{n}{3} = n^3 \times \frac{1}{27}$ steps

So, $W_{MS}(n) \geq c n^3$

$c = 1/27$

Challenge re-implement MS in $O(n^2)$ and $\Omega(n^2)$

Challenge #2 re-implement MS in $O(n) \wedge \Omega(n)$

Prove $3n^2 + 2n \in \mathcal{O}(n^2)$ ← polynomial function

Use $\mathcal{O}(n^2) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$

Choose $c =$ ___5___. Then $c \in \mathbb{R}^+$

Choose $B =$ ___0___. Then $B \in \mathbb{N}$

Assume $n \in \mathbb{N}$ # generic

Assume $n \geq B$ # antecedent

derive

some →

stuff

⋮

$f(n) \leq cn^2$

Then $n \geq B \Longrightarrow f(n) \leq cn^2$

Conclude $\forall n \in \mathbb{N}$, " "

Then $\exists B \in \mathbb{N}$, " "

Conclude $\exists c \in \mathbb{R}^+$, " "

Prove $3n^2 + 2n \in \mathcal{O}(n^2)$

Use $\mathcal{O}(n^2) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$

Then $3n^2 + 2n \leq 3n^2 + 2n^2$ # multiply by $n \geq 1$ or $n = 0$
$2n \leq 2n^2$

$= 5n^2$

$= cn^2$ # $c = 5$

Special case? what happens if you add a constant?

Prove $3n^2 + 2n + 5 \in \mathcal{O}(n^2)$

Use $\mathcal{O}(n^2) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$

structure same as previous slide, <u>but</u> we may find different $\underset{10}{\overset{\shortparallel}{c}}$, $\underset{1}{\overset{\shortparallel}{B}}$.

⋮

Then $3n^2 + 2n + 5 \leq 3n^2 + 2n^2 + 5n^2$

$2n \leq 2n^2 \; \forall n$, $n^2 \geq 1 \; \forall n \geq 1$

$5 \leq 5n^2$, $n \geq 1$

$= 10n^2$

$= cn^2$ # $c = 10$

Look at the leading term

Prove: $7n^6 - 5n^4 + 2n^3 \in \mathcal{O}(6n^8 - 4n^5 + n^2)$

Use $\mathcal{O}(6n^8 - 4n^5 + n^2) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c(6n^8 - 4n^5 + n^2)\}$

Let $c = 9/2$, Then $c \in \mathbb{R}^+$

Choose $B = 0$. Then $B \in \mathbb{N}$

Assume $n \in \mathbb{N}$ # typical natural number

assume $n \geq B$ # antecedent

Then $7n^6 - 5n^4 + 2n^3 \leq 7n^6 + 2n^3$ # $-5n^4 \leq 0$ # add $7n^6 + 2n^3$ to both sides

$\leq 7n^6 + 2n^6$ # $2n^3 \leq 2n^6$ # $\forall n \in \mathbb{N}$.

$= 9n^6 \leq 9n^8$ # $9n^6 \leq 9n^6 \cdot n^2$ # $\forall n \in \mathbb{N}$

$= c\,2n^8$ # $c = 9/2$

$= c(6n^8 - 4n^8)$ # $-4n^8 \leq -4n^5$

$\leq c(6n^8 - 4n^5) \leq c(6n^8 - 4n^5 + n^2)$

add $6n^8 - 4n^5$ to both sides of $0 \leq n^2$

Look at the leading term

Prove: $7n^6 - 5n^4 + 2n^3 \in \mathcal{O}(6n^8 - 4n^5 + n^2)$

Use $\mathcal{O}(6n^8 - 4n^5 + n^2) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c(6n^8 - 4n^5 + n^2)\}$

Then $n \geq B \Rightarrow \sim\sim\sim \leq c(\sim\sim\sim)$

Then $\forall n \in \mathbb{N}$, "__________"

introduced $\forall n$

Then $\exists B \in \mathbb{N}$, "__________"

introduce $\exists B$

Then $\exists c \in \mathbb{R}^+$, "__________"

introduced $\exists c$

how to prove $n^3 \notin \mathcal{O}(3n^2)$?

Negate $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow n^3 \leq c3n^2\}$

non-polynomials

Big-oh statements about polynomials are pretty easy to prove: $f \in \mathcal{O}(g)$ exactly when the highest-degree term of g is no smaller than the highest-degree term of f.

What about functions such as $\log(n)$ or 3^n? Logarithmic functions are in big-Oh of **any** polynomial, whereas exponential functions (with a base bigger than 1) are not in big-Oh of any polynomial. How do you prove such things?

Prove $2^n \notin \mathcal{O}(n^2)$

Use $\lim_{n\to\infty} 2^n/n^2$

Do you know anything about the ratio $2^n/n^2$, as n gets very large? How do you evaluate:

$$\lim_{n\to\infty} \frac{2^n}{n^2}$$

If the limit evaluates to ∞, then that's the same as saying:

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$

Once your enemy hands you a c, you can choose an n' with the required property.

Notes

annotated slides

- friday's annotated slides