I owe you complete infinitly many primes proof.

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Mathematical expression

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Course notes, chapter 3

Outline

non-boolean functions

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

$\lfloor x \rfloor$ *is the largest integer* $\leq x$.

Now prove the following statement (notice that we quantify over $x \in \mathbb{R}$, not $\lfloor x \rfloor \in \mathbb{R}$:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$$

Assume $x \in \mathbb{R}$ # some typical real.
Then $\lfloor x \rfloor \leq x$ # from def'n.
$< x+1$ # add x to both sides $0 < 1$
Then $\lfloor x \rfloor < x+1$ # transitivity
Conclude, $\forall x \in \mathbb{R}, \lfloor x \rfloor < x+1$ # assumed only $x \in \mathbb{R}$,
got result.

using more of the definition

You may have been disappointed that the last proof used only part of the definition of floor. Here's a symbolic re-writing of the definition of floor:

$$\forall x \in \mathbb{R} \qquad y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)$$

The full version of the definition should prove useful to prove:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$$

Scratch

$$\lfloor x \rfloor + 1 > x$$

$\in \mathbb{Z}$ $\in \mathbb{R}$

proving something false

n	0	1	2	3	4	5	6	7	8	9	10	11
a_n	0	0	1	1	2	2	3	3	4	4	5	5

Define a sequence:

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor \quad == \; n // 2$$

(of course, if you treat "/" as integer division, there's no need to take the floor. Now consider the claim:

$$S = \quad \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.

$$\neg S = \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$$

proof by cases

Sometimes your argument has to split to take into account possible properties of your generic element:

$$\forall n \in \mathbb{N}, n^2 + n \text{ is even}$$

A natural approach is to factor $n^2 + n$ as $n(n+1)$, and then consider the case where n is odd, then the case where n is even.

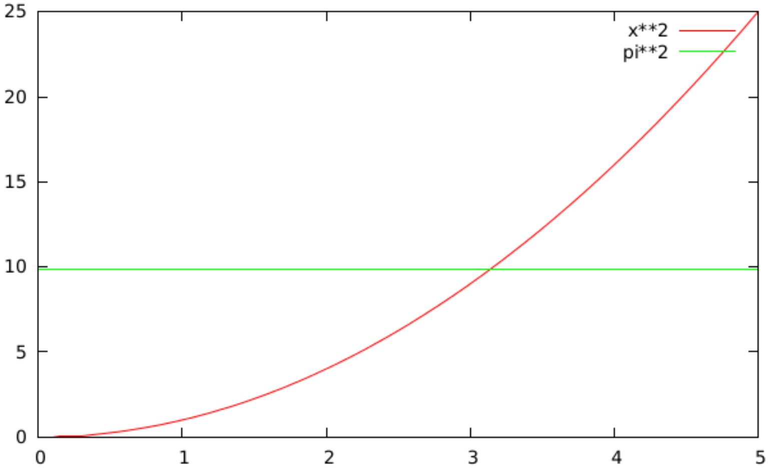
proof about limits

In proving this claim you can't control the value of e or x, but you can craft d to make things work out.

$$\forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x - \pi| < \delta \Rightarrow |x^2 - \pi^2| < e$$

The claim is true. The proof format should be already familiar to you. A good approach is to fill in as much as possible, leaving the actual value of d out until you have more intuition about it.

visualize limit proof. . .



use uniqueness

Suppose you have a predicate of the natural numbers:

$$\forall n \in \mathbb{N} \qquad S(n) \Leftrightarrow \exists k \in \mathbb{N}, n = 7k + 3$$

Is $S(3 \times 3)$ true? How do you prove that? It's useful to check out the remainder theorem from the sheet of mathematical prerequisites.

get wrong right

Be careful proving a claim false. Consider the claim, for some suitably defined X, Y and P, Q:

$$S: \qquad \forall x \in X, \forall y \in Y, P(x,y) \Rightarrow Q(x,y)$$

To **disprove** S, should you prove:

$$\forall x \in X, \forall y \in Y, P(x,y) \Rightarrow \neg Q(x,y)$$

What about

$$\forall x \in X, \forall y \in Y, \neg (P(x,y) \Rightarrow Q(x,y))$$

Explain why, or why not.

Define $T(n)$ by:

$$\forall n \in \mathbb{N} \qquad T(n) \Leftrightarrow \exists i \in \mathbb{N}, n = 7i + 1.$$

Take some scrap paper, **don't** write your name on it, and fill in as much of the proof of the following claim as possible:

$$\forall n \in \mathbb{N}, T(n) \Rightarrow T(n^2)$$

Now fill in as much of the **disproof** of the following claim as possible:

$$\forall n \in \mathbb{N}, T(n^2) \Rightarrow T(n)$$

Notes

$\lfloor x \rfloor \in \mathbb{Z} \wedge \lfloor x \rfloor \le x \wedge$
$(\forall z \in \mathbb{Z}, z \le x \Rightarrow z \le \lfloor x \rfloor)$

contrapositive
$\forall z \in \mathbb{Z}, z > \lfloor x \rfloor \Rightarrow z > x$

assume $x \in \mathbb{R}$ # generic
Then $\lfloor x \rfloor \in \mathbb{Z}$ # from defn
Then $\lfloor x \rfloor + 1 \in \mathbb{Z}$ # $\mathbb{Z}$ closed under $+$
Then $\lfloor x \rfloor + 1 > \lfloor x \rfloor$ # add $\lfloor x \rfloor$ to $1 > 0$
Then $\lfloor x \rfloor + 1 > x$ # by contrapositive
since $\lfloor x \rfloor + 1 > \lfloor x \rfloor \wedge \lfloor x \rfloor + 1 \in \mathbb{Z}$

Then $\lfloor x \rfloor > x - 1$ # subtract 1

Conclude, $\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$ # assumed only $x \in \mathbb{R}$,
got $\lfloor x \rfloor > x - 1$

Notes

$\neg S = \underline{\forall i \in \mathbb{N},} \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$

Assume i is some generic natural number.

Pick $j = \underline{i+2}$. Then $j \in \mathbb{N}$ # Since $\mathbb{N}$ closed under # plus

Then $j > i$ # add i to $2 > 0$

Then $a_j = \lfloor \frac{j}{2} \rfloor = \lfloor \frac{i+2}{2} \rfloor$

$\vdots$

$>$

$\vdots$

$\lfloor \frac{i}{2} \rfloor$

Then $\exists j \in \mathbb{N}, j > i$ and $a_j \neq a_i$ # since chosen $j \in \mathbb{N}$ # and $a_j \neq a_i$

Conclude $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$

annotated slides

- wednesday's annotated slides
- friday's annotated slides