CSC165 fall 2014

Mathematical expression

Danny Heap
heap@cs.toronto.edu
BA4270 (behind elevators)
http://www.cdf.toronto.edu/~heap/165/F14/
416-978-5899

Course notes, chapter 3

Outline

universally quantified implication, cont'd

existence

notes

annotated slides

a real inequality

Prove that for every pair of non-negative real numbers (x, y), if x is greather than y, then the geometric mean, $\sqrt{xy}$ is less than the arithmetic mean, $(x + y)/2$.

some directions work better

Prove that for any natural number n, n^2 odd implies that n is odd.

proving existence

To prove the a set is non-empty, it's enough to exhibit one element. How do you prove:

$$\exists x \in \mathbb{R}, x^3 + 3x^2 - 4x = 12$$

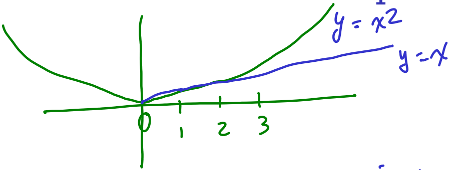
Proof

Pick $x = 2$. Then $x \in \mathbb{R}$ # well-known.

Then $x^3 + 3x^2 - 4x = 8 + 12 - 8$
$= 12$ # sub 2 for x

Then $\exists x \in \mathbb{R}, x^3 - 3x^2 - 4x = 12$ # $2 \in \mathbb{R}$ + satisfies
eqn.

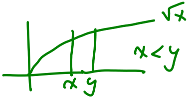
prove a claim about a sequence



$i = 1$

n	a_n
0	0
1	1
2	4
3	9
4	16
5	25
⋮	⋮

Define sequence a_n by:



$x < y \Rightarrow \sqrt{x} < \sqrt{y}$

$$\forall n \in \mathbb{N} \qquad a_n = n^2$$

Now prove:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$$

$\frac{1}{3} \leq \frac{1}{2}$

Proof

Pick $i = 2$. Then $i \in \mathbb{N}$ # $2 \in \mathbb{N}$

assume j is a representative of $\mathbb{N}$

assume $a_j \leq i$ # assume A

Then $j^2 \leq 2$ # by defn a_j & $i = 2$

Then $j \leq \sqrt{2}$ # because $\sqrt{\ }$ increasing

So $j \leq \sqrt{2} < 2$ # $\sqrt{2} \approx 1.414$

prove a claim about a sequence

Define sequence a_n by:

$$\forall n \in \mathbb{N} \qquad a_n = n^2$$

Now prove:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$$

antecedent

Cont'd.

Then, $a_j \leq i \Rightarrow j < i$ # assumed A + derived # C ⇒ consequent.

Then $\forall j \in \mathbb{N}$, # assume j was typical

$\exists i \in \mathbb{N}$, # $i = 2 \in \mathbb{N}$, + satisfies claim

contradiction — a special case of contrapositive

$F_1 \wedge F_2 \wedge \cdots \wedge F_{367893221596} \Rightarrow S$

$\neg S \Rightarrow \neg F_1 \vee \neg F_2 \vee \cdots \vee \neg F_{\ldots}$

Define the prime natural numbers as
$P = \{p \in \mathbb{N} \mid p$ has exactly two distinct divisors in $\mathbb{N}\}$. How do you prove:

$$S: \qquad \forall n \in \mathbb{N}, |P| > n$$

It would be nice to have some result R that leads to S. If you could show $R \Rightarrow S$, and that R is true, then you'd be done. But, out of many elementary results, how do you choose an R? Contradiction will often lead you there.

$\neg S \qquad \exists n \in \mathbb{N}, |P| \leq n$

non-boolean functions

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

$\lfloor x \rfloor$ *is the largest integer* $\leq x$.

Now prove the following statement (notice that we quantify over $x \in \mathbb{R}$, not $\lfloor x \rfloor \in \mathbb{R}$:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$$

using more of the definition

You may have been disappointed that the last proof used only part of the definition of floor. Here's a symbolic re-writing of the definition of floor:

$$\forall x \in \mathbb{R} \qquad y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)$$

The full version of the definition should prove useful to prove:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$$

proving something false

Define a sequence:

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor$$

(of course, if you treat "/" as integer division, there's no need to take the floor. Now consider the claim:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.

proof by cases

Sometimes your argument has to split to take into account possible properties of your generic element:

$$\forall n \in \mathbb{N}, n^2 + n \text{ is even}$$

A natural approach is to factor $n^2 + n$ as $n(n+1)$, and then consider the case where n is odd, then the case where n is even.

Notes Proof (Contradiction)

Assume $\exists n \in \mathbb{N}, |P| \leq n$

Then $\exists k \in \{0, \dots, n\}, |P| = k$

Then $\{p_1, p_2, \dots, p_k\} = P$ # just list primes

Then $m = p_1 \times p_2 \times \dots \times p_k \in \mathbb{N}$

Then $m+1 \in \mathbb{N}$. # $\mathbb{N}$ closed under $\times, +$

(added later) Then $m+1 > 1$ # $m \geq 2 \times 3 \times 5 \times \dots$

Then $\exists p \in P, p \mid (m+1)$ # every $\in \mathbb{N} > 1$ has prime factor.

also $p \mid m$ # since $m = p_1 \times p_2 \times \dots \times p_k$

So $p \mid (m+1-m) = 1$ # factor divides difference

Then $p \mid 1$ ⟶ contradiction!! ⟵

So, our assumption that $\exists n \in \mathbb{N}, |P| \leq n$ is false,

Since it leads to a contradiction

annotated slides

- monday's annotated slides
- wednesday's annotated slides
- friday's annotated slides