CSC165 fall 2014

Mathematical expression

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Course notes, chapter 4

Outline

Bounded below

some theorems

notes

how to prove $n^3 \notin \mathcal{O}(3n^2)$?

Negate $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow n^3 \leq c3n^2$ ✓

$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n^3 > c3n^2$

assume $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$ # to introduce $\forall$

Choose $n = \underline{B + \lceil 3c \rceil + 1}$. Then $n \in \mathbb{N}$. # $\mathbb{N}$ closed under +

Then $n \geq B$ # added something ≥ 0 to B

Then $n^3 = n \cdot n^2$

Then $n^3 > c3n^2$ # $n > 3c$, $n \geq \lceil 3c \rceil + 1$

Then $n \geq B \wedge n^3 > c3n^2$

⋮

Conclude $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n^3 > c3n^2$

(That is, $n^3 \notin \mathcal{O}(3n^2)$)

non-polynomials

Big-oh statements about polynomials are pretty easy to prove: $f \in \mathcal{O}(g)$ exactly when the highest-degree term of g is no smaller than the highest-degree term of f.

What about functions such as $\log(n)$ or 3^n? Logarithmic functions are in big-Oh of **any** polynomial, whereas exponential functions (with a base bigger than 1) are not in big-Oh of any polynomial. How do you prove such things?

Prove $2^n \notin \mathcal{O}(n^2)$

Use $\lim_{n\to\infty} 2^n/n^2$

Do you know anything about the ratio $2^n/n^2$, as n gets very large? How do you evaluate:

$$\lim_{n\to\infty} \frac{2^n}{n^2}$$

If the limit evaluates to ∞, then that's the same as saying:

$$\forall c \in \mathbb{R}^+, \exists n_c \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_c \Rightarrow \frac{2^n}{n^2} > c$$

Once your enemy hands you a c, you can choose an n_c with the required property.

prove $2^n \notin \mathcal{O}(n^2)$, that is $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > cn^2$

Using l'Hôpital's rule and limits

i.e. $\frac{2^n}{n^2} > c \longleftrightarrow 2^n > cn^2$

Assume $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$ # generic

Then $\exists n_c \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_c \Rightarrow 2^n/n^2 > c$

Pick $n' = n_c$

Pick $n = n' + B$. Then $n \in \mathbb{N}$.

Then $n \geq B$ # $n' + B \geq B$, $n' \in \mathbb{N}$.

Then $\frac{2^n}{n^2} > c$, so $2^n > cn^2$ # by choice of n

Then $n \geq B \wedge 2^n > cn^2$

Conclude $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > cn^2$

(That is, $2^n \notin \mathcal{O}(n^2)$)

big-Omega

Notice that the definition of big-Omega differs in just one character from big-Oh:

$$\Omega(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \geq cg(n)\}$$

The rôle of B is, as with big-Oh, to act as a breakpoint, so comparisons don't have to start at the origin.

The rôle of c is to scale g *down* below f.

If you're proving $f \in \Omega(g)$, you get to choose c and B to suit your proof. Notice that it would be really unfair to allow c to be zero.

one more bound

It often happens that functions are bounded above *and* below by the same function. In other words, $f \in \mathcal{O}(G) \wedge f \in \Omega(g)$. We combine these two concepts into $f \in \Theta(g)$.

$$\Theta(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N},$$
$$\forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

You might want to draw pictures, and conjecture about appropriate values of c_1, c_2, B for $f = 5n^2 + 15$ and $g = n^2$.

How to prove general statement about two functions:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g, h \in \mathcal{F}, (f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)) \Rightarrow f \in \mathcal{O}(h)$

how about:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \Rightarrow g \in \Omega(f)$

prove or disprove:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \Rightarrow f \cdot f \in \mathcal{O}(g \cdot g)$

prove or disprove:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g \in \mathcal{F}, (f \in \mathcal{O}(h) \wedge g \in \mathcal{O}(h)) \Rightarrow (f + g) \in \mathcal{O}(h)$

prove or disprove:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \Rightarrow f \in \mathcal{O}(g \cdot g)$

Notes