# CSC165 Fall 2014, Assignment #3

## Sample Solutions

1. **Prove or disprove:**

$$\forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall x, y \in \mathbb{R}^+, |x - y| > d \Rightarrow |x + y| > e$$

**Sample solution:** This claim is true. The intuition is that, first, it should be true that $|x - y| < |x + y|$ for two positive numbers $x$ and $y$; then, if we pick $d = e$, then the smaller number ($|x - y|$) being larger than $e$ would imply that the larger number ($|x + y|$) must also be larger than $e$.

**Proof:**

Assume $e \in \mathbb{R}^+$ # generic real number

  Pick $d = e$, then $d \in \mathbb{R}^+$ # since $e \in \mathbb{R}^+$

    Assume $x, y \in \mathbb{R}^+$ and that $|x - y| > d$ # generic real numbers, and the antecedent

      then $(x - y \geq 0) \vee (x - y < 0)$ # only two possible cases

      Case 1: assume $x - y \geq 0$

        then $|x - y| = x - y < x - y + 2y = x + y = |x + y|$ # $2y > 0$ and $x + y > 0$

        then $|x - y| < |x + y|$ # transitivity

      Case 2: assume $x - y < 0$

        then $y - x > 0$ # multiple both sides by $-1$

        then $|x - y| = y - x < y - x + 2x = x + y = |x + y|$ # $2x > 0$ and $x + y > 0$

        then $|x - y| < |x + y|$ # transitivity

      then $|x - y| < |x + y|$ # since it's true for both cases

      then $|x + y| > |x - y|$ # reverse inequality

      then $|x - y| > e$ # since we picked $d = e$, and $|x - y| > d$

      then $|x + y| > e$ # transitivity, $|x + y| > |x - y| > e$

    then $\forall x, y \in \mathbb{R}^+, |x - y| > d \Rightarrow |x + y| > e$ # introduce $\forall$ and implication

  then $\exists d \in \mathbb{R}^+, \forall x, y \in \mathbb{R}^+, |x - y| > d \Rightarrow |x + y| > e$ # introduce $\exists$

then $\forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall x, y \in \mathbb{R}^+, |x - y| > d \Rightarrow |x + y| > e$ # introduce $\forall$

**Alternative ways of proving** $|x - y| < |x + y|$

(a) Using triangle inequality:

$|x - y| = |x + (-y)|$

  $\leq |x| + |-y|$ # triangle inequality

  $= x + y$ # since $x, y > 0$, and definition of absolute value

  $= |x + y|$ # since $x + y > 0$

then $|x - y| \leq |x + y|$ # not as strong as "$<$", but good enough for this question.

(b) Compare the square of both sides:

then $-2xy < 2xy$ # $-2 < 2$ and $xy > 0$
then $x^2 + y^2 - 2xy < x^2 + y^2 + 2xy$ # add $x^2 + y^2$ to both sides
then $(x - y)^2 < (x + y)^2$ # algebra
then $|x - y| < |x + y|$ # square root of both sides

2. **Prove or disprove**: $6n^3 - 4n^2 + 3n + 2$ is in $\Omega(5n^3 - n^2 + n + 1)$.

**Sample solution:** This claim is true, since the highest degrees of both polynomials are the same. By definition of $\Omega$, we need to prove the following statement:

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 6n^3 - 4n^2 + 3n + 2 \geq c \cdot (5n^3 - n^2 + n + 1)$$

Use the chain of overestimate/underestimate to find proper $c$ and $B$.

**Proof:**

Pick $c = 2/7, B = 1$, then $c \in \mathbb{R}^+, B \in \mathbb{N}$ # in order to introduce $\exists$

Assume $n \in \mathbb{N}$ and $n \geq B$ # generic natural number and the antecedent

then $6n^3 - 4n^2 + 3n + 2 > 6n^3 - 4n^2$ # remove positive term $3n + 2$

$\geq 6n^3 - 4n^2 \times n = 2n^3$# multiply a negative term by $n \geq B = 1$

$= (2/7) \cdot (7n^3) = c \cdot (7n^3)$ # we picked $c = 2/7$

$= c \cdot (5n^3 + n^3 + n^3)$ # $7 = 5 + 1 + 1$

$\geq c \cdot (5n^3 + n + 1)$ # $n^3 \geq n, n^3 \geq 1$ since $n \geq B = 1$

$> c \cdot (5n^3 - n^2 + n + 1)$ # add a negative term $-n^2$

then $6n^3 - 4n^2 + 3n + 2 \geq c \cdot (5n^3 - n^2 + n + 1)$ # transitivity

then $\forall n \in \mathbb{N}, n \geq B \Rightarrow 6n^3 - 4n^2 + 3n + 2 \geq c \cdot (5n^3 - n^2 + n + 1)$ # introduce $\forall, \Rightarrow$

then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 6n^3 - 4n^2 + 3n + 2 \geq c \cdot (5n^3 - n^2 + n + 1)$ # intro $\exists$

then $6n^3 - 4n^2 + 3n + 2 \in \Omega(5n^3 - n^2 + n + 1)$ # by definition of $\Omega$

3. **Prove or disprove**: $15n^2$ is in $\Omega(3 \times 2^n)$. **Hint**: Consider using limit techniques from calculus, including l'Hôpital's rule as part of this proof. Please talk to your TA/instructor/Help Centre when needed.

**Sample solution**: This claim is False. The polynomial $n^2$ grows much slower than the exponential $2^n$ therefore cannot be lower-bounded by the exponential. So we need to prove $15n^2 \notin \Omega(3 \times 2^n)$, which, by the negation of the definition of $\Omega$, is

$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, (n \geq B) \wedge (15n^2 < c \cdot (3 \times 2^n))$$

**Proof:**

By limit technique in calculus

$$\lim_{n\to\infty} \frac{15n^2}{3 \cdot 2^n} = \lim_{n\to\infty} \frac{(15n^2)'}{(3 \cdot 2^n)'} = \lim_{n\to\infty} \frac{30n}{3\ln 2 \cdot 2^n} = \lim_{n\to\infty} \frac{(30n)'}{(3\ln 2 \cdot 2^n)'}$$
$$= \lim_{n\to\infty} \frac{30}{3\ln 2 \ln 2 \cdot 2^n} = 0 \text{ \# apply l'Hôpital's rule twice}$$

then $\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow (15n^2)/(3 \cdot 2^n) < c$ # definition of limit

Assume $c \in \mathbb{R}^+, B \in \mathbb{N}$ # in order to introduce $\forall$

then $\exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow (15n^2)/(3 \cdot 2^n) < c$ # by the definition of the limit

Pick $n = \max(n', B)$, then $n \in \mathbb{N}$ # $n', B \in \mathbb{N}$

then $n \geq n'$ # by definition of max

then $(15n^2)/(3 \cdot 2^n) < c$ # since $n \geq n' \Rightarrow (15n^2)/(3 \cdot 2^n) < c$

then $15n^2 < c \cdot (3 \cdot 2^n)$ # multiply both sides by $3 \cdot 2^n > 0$

also $n \geq B$ # by definition of max

then $(n \geq B) \wedge (15n^2 < c \cdot (3 \cdot 2^n)$ # conjunction introduction

then $\exists n \in \mathbb{N}, (n \geq B) \wedge (15n^2 < c \cdot (3 \times 2^n))$ # introduce $\exists$

then $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, (n \geq B) \wedge (15n^2 < c \cdot (3 \times 2^n))$ # introduce $\forall$

then $15n^2 \notin \Omega(3 \times 2^n)$ # by the negation of the definition of $\Omega$

4. **Prove or disprove:** $2^n$ is in $\mathcal{O}(3^n)$. **Hint:** Consider using the limit techniques of calculus and notice that

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n$$

**Sample solution:** This claim is True. By definition of $\mathcal{O}$, we need to prove the following

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 2^n \leq c \cdot (3^n)$$

**Proof:**

By limit technique in calculus

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

then $\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow 2^n/3^n < c$ # definition of limit

Pick $c = 165.42$, then $c \in \mathbb{R}^+$ # any $c \in \mathbb{R}^+$ is okay

then $\exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow 2^n/3^n < c$ # by definition of the limit

Pick $B = n'$, then $B \in \mathbb{N}$ # $n' \in \mathbb{N}$

Assume $n \in \mathbb{N}, n \geq B$ # in order to introduce $\forall$ and $\Rightarrow$

then $n \geq n'$ # since $B = n'$

then $2^n/3^n < c$ # $n \geq n' \Rightarrow 2^n/3^n < c$

then $2^n < c \cdot 3^n$ # multiply both sides by $3^n$

then $\forall n \in \mathbb{N}, n \geq B \Rightarrow 2^n \leq c \cdot (3^n)$ # introduce $\forall$ and $\Rightarrow$

then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 2^n \leq c \cdot (3^n)$ # introduce $\exists$

then $2^n \in \mathcal{O}(3^n)$ # definition of $\mathcal{O}$

5. Given the set $\mathcal{F} = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0}\}$, **Prove or disprove:**

$$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \vee f \in \Omega(g)$$

**Sample solution:** This claim is False. To diprove it, we should prove the following negated statement

$$\exists f, g \in \mathcal{F}, f \notin \mathcal{O}(g) \wedge f \notin \Omega(g)$$

*i.e.*, we need to find an example pair of functions $f, g$ such that $g$ is neither an upper bound nor a lower bound of $f$. Intuitively, we need two functions that are "intertwined" with each other so that no one is constantly bounding the other, at least one of them should be "oscillating" in some manner. Here are a few example pairs that would work:

$$f(n) = n \mod 2 \qquad g(n) = (n+1) \mod 2$$
$$f(n) = 1 + \cos(\pi n) \qquad g(n) = 1 - \cos(\pi n)$$
$$f(n) = 1 + \cos(\pi n) \qquad g(n) = \frac{1}{n+1}$$

Note that the functions we choose must satisfy $f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}$, *i.e.*, a non-negative function with natural number inputs. Here we write the proof for the first example.

**Proof:**

Pick $f(n) = n \mod 2$ and $g(n) = (n+1) \mod 2$, then $f, g \in \mathcal{F}$ # $\mathbb{N} \mapsto \mathbb{R}^{\geq 0}$

Assume $c \in \mathbb{R}^+, B \in \mathbb{N}$ # in order to introduce $\forall$
Pick $n = 2B + 1$, then $n \in \mathbb{N}$ # $B, 1, 2 \in \mathbb{N}$
then $n \geq B$ # add $B$ to both sides of $B + 1 \geq 0$
then $n$ is odd # by definition of odd
then $f(n) = n \mod 2 = 1$ # $n$ is odd
then $g(n) = (n+1) \mod 2 = 0$ # $n+1$ is even
then $f(n) > c \cdot g(n)$ # $1 > 0 = c \cdot 0$
then $(n \geq B) \wedge (f(n) > c \cdot g(n))$ # conjunction introduction
then $\exists n \in \mathbb{N}, (n \geq B) \wedge (f(n) > c \cdot g(n))$ # introduce $\exists$
then $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, (n \geq B) \wedge (f(n) > c \cdot g(n))$ # introduce $\forall$
then $f \notin \mathcal{O}(g)$ # negation of definition of $\mathcal{O}$

Assume $c \in \mathbb{R}^+, B \in \mathbb{N}$ # in order to introduce $\forall$
Pick $n = 2B$, then $n \in \mathbb{N}$ # $B, 2 \in \mathbb{N}$
then $n \geq B$ # add $B$ to both sides of $B \geq 0$
then $n$ is even # by definition of even
then $f(n) = n \mod 2 = 0$ # $n$ is even
then $g(n) = (n+1) \mod 2 = 1$ # $n+1$ is odd
then $f(n) < c \cdot g(n)$ # $0 < c \cdot 1$ since $c > 0$
then $(n \geq B) \wedge (f(n) < c \cdot g(n))$ # conjunction introduction
then $\exists n \in \mathbb{N}, (n \geq B) \wedge (f(n) < c \cdot g(n))$ # introduce $\exists$
then $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, (n \geq B) \wedge (f(n) < c \cdot g(n))$ # introduce $\forall$
then $f \notin \Omega(g)$ # negation of definition of $\Omega$

then $\exists f, g \in \mathcal{F}, f \notin \mathcal{O}(g) \wedge f \notin \Omega(g)$ # introduce $\exists$

6. Prove that the function `meaning_of_life` below is not computable:

```
def meaning_of_life(f, I) :
    """
    Return True if f(I) returns 42, False otherwise.
    """
```

Emulate the technique from the course notes to reduce `halt` to `meaning_of_life`

**Sample solution:** We simply emulate the proof for Claim 5.1 in the course notes. Reduce `halt` to `meaing_of_life`, and reach a contradiction by assuming `meaing_of_life` is computable.

**Proof:**

For a contradiction, assume that `meaing_of_life` is computable, *i.e.*, it can be implemented as a Python function.

Then consider the following program.

```
def halt(f, I):

    def f_prime(x):
        # Use the f and I that are passed to halt
        # x is ignored
        f(I)
        return 42

    return meaning_of_life(f_prime, I)
```

If `f(I)` halts, then `f_prime(I)` will return 42, so `meaing_of_life(f_prime, I)` returns `True` and `halt(f, I)` returns `True`.

If `f(I)` does not halt, then `f_prime(I)` will never return anything, *i.e.*, not returning 42, so `meaing_of_life(f_prime, I)` returns `False` and `halt(f, I)` returns `False`.

then this program is a correct implementation of `halt`, and since `meaing_of_life` is computable (as assumed), then `halt` is computable.

then contradiction because we know for a fact that `halt` is NOT computable.

then `meaing_of_life` is NOT computable.