CSC165 Fall 2014, Assignment #2

sample solutions

1. For $x \in \mathbb{R}$, define $\lfloor x \rfloor$ by:

$$\lfloor x \rfloor \in \mathbb{Z} \wedge \lfloor x \rfloor \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq \lfloor x \rfloor).$$

... where $\mathbb{Z}$ stands for the set of integers, and $\mathbb{R}$ stands for the set of real numbers. Use the definition of $\lfloor x \rfloor$ to prove or disprove each of the following claims, using the structured proof technique from this course. **Note:** You must use the definition given here, not some other (possibly equivalent) definition for $\lfloor x \rfloor$.

Claim 1.1:

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor.$$

Sample solution: This claim is true. It says if a real number x is larger than another real number y, then x's floor cannot be smaller than y's floor. We present two proofs here, one directly uses the definition, the other uses contradiction.

Proof directly using definition:

Assume $x \in \mathbb{R}, y \in \mathbb{R}$ # generic real numbers
  Assume $x > y$ # the antecedent
    then $y < x$ # reverse the inequality
    and $\lfloor y \rfloor \leq y$ # by definition of $\lfloor y \rfloor$
    then $\lfloor y \rfloor \leq x$ # transitivity of inequality, $\lfloor y \rfloor \leq y < x$
    and $\lfloor y \rfloor \in \mathbb{Z}$ # by definition of $\lfloor y \rfloor$
    then $\lfloor y \rfloor \leq \lfloor x \rfloor$ # by definition of $\lfloor x \rfloor$
    then $\lfloor x \rfloor \geq \lfloor y \rfloor$ # reverse the inequality
  then $x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor$ # introduce antecedent
then $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor$ # introduce $\forall$

Proof by contradiction:

Assume $\neg(\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor)$ # for the sake of contradiction
  then $\exists x \in \mathbb{R}, y \in \mathbb{R}, (x > y) \wedge (\lfloor x \rfloor < \lfloor y \rfloor)$ # the negation
  Let $x_0 \in \mathbb{R}, y_0 \in \mathbb{R}$ be such that $(x_0 > y_0) \wedge (\lfloor x_0 \rfloor < \lfloor y_0 \rfloor)$
  then $\lfloor x_0 \rfloor < \lfloor y_0 \rfloor$ # conjunction elimination

and $\lfloor x_0 \rfloor \in \mathbb{Z}, \lfloor y_0 \rfloor \in \mathbb{Z}$ # by definition of floor
then $\lfloor x_0 \rfloor + 1 \leq \lfloor y_0 \rfloor$ # the smallest possible difference between two distinct integers is 1
then $\lfloor x_0 \rfloor + 1 \leq y_0$ # since $\lfloor y_0 \rfloor \leq y_0$ by definition of $\lfloor y_0 \rfloor$
then $\lfloor x_0 \rfloor + 1 < x_0$ # since $y_0 < x_0$ as how x_0 and y_0 are picked
and $\lfloor x_0 \rfloor + 1 \in \mathbb{Z}$ # $\lfloor x_0 \rfloor \in \mathbb{Z}$ and $1 \in \mathbb{Z}$
then $\lfloor x_0 \rfloor + 1 \leq \lfloor x_0 \rfloor$ # by definition of $\lfloor x_0 \rfloor$ that $\forall z \in \mathbb{Z}, z \leq x_0 \Rightarrow z \leq \lfloor x_0 \rfloor$
then $1 \leq 0$ # subtract $\lfloor x_0 \rfloor$ from both sides, and contradiction with that $1 > 0$
then $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor$ # negation of assumption because of contradiction

Claim 1.2:

$$\forall x \in \mathbb{R}, \forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall w \in \mathbb{R}, |x - w| < d \Rightarrow |\lfloor x \rfloor - \lfloor w \rfloor| < e$$

Sample solution: This claim is false. Intuitively, this claim says for all x, when w is getting really really close to x then $\lfloor w \rfloor$ gets arbitrarily close to $\lfloor x \rfloor$. This is not quite true because, for example, 3.9999 is really close to 4 but $\lfloor 3.9999 \rfloor = 3$ is not that close at all to $\lfloor 4 \rfloor = 4$. So we will prove the negation of this claim which is

$$\exists x \in \mathbb{R}, \exists e \in \mathbb{R}^+, \forall d \in \mathbb{R}^+, \exists w \in \mathbb{R}, (|x - w| < d) \wedge (|\lfloor x \rfloor - \lfloor w \rfloor| \geq e)$$

Proof:
Pick $x = 4$, $e = 0.5$, then $x \in \mathbb{R}$ and $e \in \mathbb{R}^+$
Assume $d \in \mathbb{R}^+$ # a generic positive real number
Pick $w = 4 - 0.9d$
then $|x - w| = |4 - (4 - 0.9d)| = 0.9d < d$
and $w < 4$ # add 4 to both sides of $-0.9d < 0$
then $\lfloor w \rfloor \leq 3$ # $\lfloor w \rfloor \leq w < 4$
also $\lfloor x \rfloor = \lfloor 4 \rfloor = 4$
then $\lfloor x \rfloor - \lfloor w \rfloor \geq 4 - 3 = 1 \geq 0.5 = e$ # $-\lfloor w \rfloor \geq -3$
then $\lfloor x \rfloor - \lfloor w \rfloor \geq e$ # transitivity of inequality
then $|\lfloor x \rfloor - \lfloor w \rfloor| \geq e$ # absolute value of a positive number
then $(|x - w| < d) \wedge (|\lfloor x \rfloor - \lfloor w \rfloor| \geq e)$ # conjunction introduction
then $\forall d \in \mathbb{R}^+, \exists w \in \mathbb{R}, (|x - w| < d) \wedge (|\lfloor x \rfloor - \lfloor w \rfloor| \geq e)$ # introduce $\forall$
then $\exists x \in \mathbb{R}, \exists e \in \mathbb{R}^+, \forall d \in \mathbb{R}^+, \exists w \in \mathbb{R}, (|x - w| < d) \wedge (|\lfloor x \rfloor - \lfloor w \rfloor| \geq e)$ # introduce $\exists$

Claim 1.3:

$$\exists x \in \mathbb{R}, \forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall w \in \mathbb{R}, |x - w| < d \Rightarrow |\lfloor x \rfloor - \lfloor w \rfloor| < e$$

Sample solution: This claim is true. It says that there exists a point x such that when w gets really close to x, $\lfloor w \rfloor$ gets arbitrarily close to $\lfloor x \rfloor$. From the previous proof, we learned that points like $x = 4$ are not good examples because the floor function is not continuous (or, is "jumping") at these points; however, any other points except these "jumping" points would be continuous and valid examples for this claim, such as $x = 4.5$.

Proof:

Pick $x = 4.5$, then $x \in \mathbb{R}$

Assume $e \in \mathbb{R}^+$ # a generic positive real number

Pick $d = 0.49$, then $d \in \mathbb{R}^+$

Assume $w \in \mathbb{R}$ # a generic real number

Assume $|x - w| < d$ # assume the antecedent of the $\Rightarrow$

then $-d < w - x < d$ # $|x| < a \Leftrightarrow -a < x < a$

then $x - d < w < x + d$

then $4.01 < w < 4.99$ # $x = 4.5, d = 0.49$

then $\lfloor w \rfloor = 4$ # by definition of floor

and $\lfloor x \rfloor = \lfloor 4.5 \rfloor = 4$ # $x = 4.5$ as picked and definition of floor

then $|\lfloor x \rfloor - \lfloor w \rfloor| = |4 - 4| = 0$

then $|\lfloor x \rfloor - \lfloor w \rfloor| < e$ # $e \in \mathbb{R}^+$

then $|x - w| < d \Rightarrow |\lfloor x \rfloor - \lfloor w \rfloor| < e$ # introduce antecedent

then $\forall w \in \mathbb{R}, |x - w| < d \Rightarrow |\lfloor x \rfloor - \lfloor w \rfloor| < e$ # introduce $\forall$

then $\exists d \in \mathbb{R}^+, \forall w \in \mathbb{R}, |x - w| < d \Rightarrow |\lfloor x \rfloor - \lfloor w \rfloor| < e$ # introduce $\exists$

then $\forall e \in \mathbb{R}^+ \exists d \in \mathbb{R}^+, \forall w \in \mathbb{R}, |x - w| < d \Rightarrow |\lfloor x \rfloor - \lfloor w \rfloor| < e$ # introduce $\forall$

then $\exists x \in \mathbb{R}, \forall e \in \mathbb{R}^+ \exists d \in \mathbb{R}^+, \forall w \in \mathbb{R}, |x - w| < d \Rightarrow |\lfloor x \rfloor - \lfloor w \rfloor| < e$ # introduce $\exists$

Claim 1.4:

$$\exists x \in \mathbb{R}, \lfloor x + 1 \rfloor \neq \lfloor x \rfloor + 1$$

Sample solution: This claim is false. We will prove the negation of this statement which is.

$$\forall x \in \mathbb{R}, \lfloor x + 1 \rfloor = \lfloor x \rfloor + 1$$

We will prove the equality $a = b$ by proving $(a \leq b) \wedge (b \leq a)$. The key is to make wise use of the definition of the floor, especially the "$z \leq x \Rightarrow z \leq \lfloor x \rfloor$" part.

Proof:

Assume $x \in \mathbb{R}$ # a generic real number x

then $\lfloor x \rfloor \leq x$ # by definition of $\lfloor x \rfloor$

then $\lfloor x \rfloor + 1 \leq x + 1$ # add 1 to both sides

then $\lfloor x \rfloor + 1 \leq \lfloor x + 1 \rfloor$ # $\lfloor x \rfloor + 1 \in \mathbb{Z}$ and by definition of $\lfloor x + 1 \rfloor$

also $\lfloor x + 1 \rfloor \leq x + 1$ # by definition of $\lfloor x + 1 \rfloor$

then $\lfloor x + 1 \rfloor - 1 \leq x$ # subtract 1 from both sides

then $\lfloor x + 1 \rfloor - 1 \leq \lfloor x \rfloor$ # $\lfloor x + 1 \rfloor - 1 \in \mathbb{Z}$ and by definition of $\lfloor x \rfloor$

then $\lfloor x + 1 \rfloor \leq \lfloor x \rfloor + 1$ # add 1 to both sides

then $(\lfloor x \rfloor + 1 \leq \lfloor x + 1 \rfloor) \wedge (\lfloor x + 1 \rfloor \leq \lfloor x \rfloor + 1)$ # conjunction introduction

then $\lfloor x + 1 \rfloor = \lfloor x \rfloor + 1$ # $(a \leq b \wedge b \leq a) \Leftrightarrow a = b$

then $\forall x \in \mathbb{R}, \lfloor x + 1 \rfloor = \lfloor x \rfloor + 1$ # introduce $\forall$

2. **Prove** or **disprove** the claim, and **prove** or **disprove** the converse:

Claim 2.1:

$$\forall n \in \mathbb{N}, (\exists k \in \mathbb{N}, n = 5k + 2) \Rightarrow \left(\exists j \in \mathbb{N}, n^2 = 5j + 4\right)$$

Sample solution: The original claim is true, the proof is similar to what we did in the lectures and tutorials. The converse of this claim is false, because $n = 3, n^2 = 9$ would be a counter-example.

Proof of the original:

Assume $n \in \mathbb{N}$ # a generic natural number

Assume $\exists k \in \mathbb{N}, n = 5k + 2$ # the antecedent

Let $k_0 \in \mathbb{N}$ be such that $n = 5k_0 + 2$

then $n^2 = (5k_0 + 2)^2 = 25k_0^2 + 20k_0 + 4 = 5(5k_0^2 + 4k_0) + 4$

Let $j = 5k_0^2 + 4k_0$, then $j \in \mathbb{N}$ # $k_0, 5, 4 \in \mathbb{N}$

then $n^2 = 5j + 4$

then $\exists j \in \mathbb{N}, n^2 = 5j + 4$

then $(\exists k \in \mathbb{N}, n = 5k + 2) \Rightarrow (\exists j \in \mathbb{N}, n^2 = 5j + 4)$ # introduce antecedent

then $\forall n \in \mathbb{N}, (\exists k \in \mathbb{N}, n = 5k + 2) \Rightarrow (\exists j \in \mathbb{N}, n^2 = 5j + 4)$ # introduce $\forall$

The negation of the converse of the claim is

$$\exists n \in \mathbb{N}, (\exists j \in \mathbb{N}, n^2 = 5j + 4) \wedge \neg (\exists k \in \mathbb{N}, n = 5k + 2)$$

Proof of the negation of converse:

Pick $n = 3$, then $n \in \mathbb{N}$

then $n^2 = 9 = 5 \times 1 + 4$

then $\exists j \in \mathbb{N}, n^2 = 5j + 4$ # $1 \in \mathbb{N}$

also $n = 3 = 5 \times 0 + 3$

then $\neg (\exists k \in \mathbb{N}, n = 5k + 2)$ # uniqueness of remainder

then $(\exists j \in \mathbb{N}, n^2 = 5j + 4) \wedge \neg (\exists k \in \mathbb{N}, n = 5k + 2)$ # conjunction introduction

then $\exists n \in \mathbb{N}, (\exists j \in \mathbb{N}, n^2 = 5j + 4) \wedge \neg (\exists k \in \mathbb{N}, n = 5k + 2)$ # introduce $\exists$

Claim 2.2:

$$\forall m, n \in \mathbb{N}, (\exists k \in \mathbb{N}, m = 7k + 3) \wedge (\exists j \in \mathbb{N}, n = 7j + 4) \Rightarrow (\exists i \in \mathbb{N}, mn = 7i + 5)$$

Sample solution: The original claim is true. The converse of this claim is false, because we can easily find a counter-example such as $m = 1, n = 5$.

Proof of the original:

Assume $m, n \in \mathbb{N}$ # two generic natural numbers

Assume $(\exists k \in \mathbb{N}, m = 7k + 3) \wedge (\exists j \in \mathbb{N}, n = 7j + 4)$ # the antecedent

Let $k_0 \in \mathbb{N}$ be such that $m = 7k_0 + 3$, and $j_0 \in \mathbb{N}$ be such that $n = 7j_0 + 4$

then $mn = (7k_0 + 3)(7j_0 + 4) = 49k_0j_0 + 28k_0 + 21j_0 + 12$

$= 7(7k_0j_0 + 4k_0 + 3j_0 + 1) + 5$

Let $i = 7k_0j_0 + 4k_0 + 3j_0 + 1$

then $mn = 7i + 5$

then $\exists i \in \mathbb{N}, mn = 7i + 5$

then $(\exists k \in \mathbb{N}, m = 7k + 3) \wedge (\exists j \in \mathbb{N}, n = 7j + 4) \Rightarrow (\exists i \in \mathbb{N}, mn = 7i + 5)$

then $\forall m, n \in \mathbb{N}, (\exists k \in \mathbb{N}, m = 7k + 3) \wedge (\exists j \in \mathbb{N}, n = 7j + 4) \Rightarrow (\exists i \in \mathbb{N}, mn = 7i + 5)$

The negation of the converse of the claim is

$$\exists m, n \in \mathbb{N}, (\exists i \in \mathbb{N}, mn = 7i + 5) \wedge [\neg(\exists k \in \mathbb{N}, m = 7k + 3) \vee \neg(\exists j \in \mathbb{N}, n = 7j + 4)]$$

Proof of the negation of converse:

Pick $m = 1, n = 5$, then $m, n \in \mathbb{N}$

then $mn = 1 \times 5 = 5 = 7 \times 0 + 5$

then $\exists i \in \mathbb{N}, mn = 7i + 5$ # $0 \in \mathbb{N}$

also $m = 1 = 7 \times 0 + 1$

then $\neg(\exists k \in \mathbb{N}, m = 7k + 3)$ # uniqueness of remainder

then $\neg(\exists k \in \mathbb{N}, m = 7k + 3) \vee \neg(\exists j \in \mathbb{N}, n = 7j + 4)$ # disjunction introduction

then $(\exists i \in \mathbb{N}, mn = 7i + 5) \wedge [\neg(\exists k \in \mathbb{N}, m = 7k + 3) \vee \neg(\exists j \in \mathbb{N}, n = 7j + 4)]$

then $\exists m, n \in \mathbb{N}, (\exists i \in \mathbb{N}, mn = 7i + 5) \wedge [\neg(\exists k \in \mathbb{N}, m = 7k + 3) \vee \neg(\exists j \in \mathbb{N}, n = 7j + 4)]$
introduce $\exists$