CSC165 Fall 2014, Assignment #1
sample solutions

1. Suppose A is the set of acronyms, $B(a)$ means a is bifurcated, $C(a)$ means a is catheterized, and $D(a)$ means a is diagonal (you do not need to worry about the meaning of acronym, bifurcated, catheterized, or diagonal for the rest of this question). Write the negation of each of the following statements in English, and then in symbolic form.

   (a) All acronyms are catheterized unless they are bifurcated.

   Solution: Some acronym is neither catheterized nor bifurcated.

   $$\exists a \in A, \neg C(a) \wedge \neg B(a)$$

   (b) There are no acronyms that are both bifurcated and diagonal.

   Solution: Some acronym is both bifurcated and diagonal.

   $$\exists a \in A, B(a) \wedge D(a)$$

   (c) All non-diagonal acronyms are catheterized.

   Solution: Some acronym that is not diagonal is also not catheterized.

   $$\exists a \in A, \neg D(a) \wedge \neg C(a)$$

   (d) There are diagonal acronyms that are bifurcated.

   Solution: No diagonal acronym is bifurcated.

   $$\forall a \in A, D(a) \Rightarrow \neg B(a)$$

   (e) For an acronym to be catheterized, it is necessary and sufficient that it be diagonal.

   Solution: Some acronym is either catheterized and not diagonal, or diagonal and not catheterized.

   $$\exists a \in A, (C(a) \wedge \neg D(a)) \vee (\neg C(a) \wedge D(a))$$

2. Consider the sentence:

   S: Every employee who is honest and persistent is successful or bored.

   Each of the statements below is equivalent to either the converse, the contrapositive, or the negation of S. You must decide which label fits each statement, and explain your thinking.

Solution (preface): Recall that the contrapositive is equivalent to S itself. Here's S expressed symbolically, with E being the set of employees, $H(x)$ meaning x is honest, $P(x)$ meaning x is persistent, $S(x)$ meaning x is successful and $B(x)$ meaning x is bored:

$$\forall x \in E, (H(x) \wedge P(x)) \Rightarrow (S(x) \vee B(x))$$

(a) All employees who are successful or bored must be honest and persistent.

Solution: Written symbolically, this is:

$$\forall x \in E, (S(x) \vee B(x)) \Rightarrow (H(x) \wedge P(x))$$

This is the converse of S since it is only reversing the direction of the implication sign.

(b) Every employee who is neither successful nor bored is either dishonest or not persistent.

Solution: Written symbolically, this is:

$$\forall x \in E, (\neg S(x) \wedge \neg B(x)) \Rightarrow (\neg H(x) \vee \neg P(x))$$

This is the contrapositive of S since its antecedent is the negation of S's consequent (using De Morgan's Law), and its consequent is the negation of S's antecedent (again using De Morgan's Law).

(c) Some honest and persistent employee is neither successful nor bored.

Solution: Written symbolically, this is:

$$\exists x \in E, (H(x) \wedge P(x)) \wedge (\neg S(x) \wedge \neg B(x))$$

This is the negation of S, since the negation of universal yields the existential, and the negation of implication follows this rule: $\neg(P \Rightarrow Q) \Leftrightarrow P \wedge \neg Q$.

(d) All employees who are dishonest or not persistent must be unsuccessful and not bored.

Solution: Written symbolically, this is:

$$\forall x \in E, (\neg H(x) \vee \neg P(x)) \Rightarrow (\neg S(x) \wedge \neg B(x))$$

This is the converse of the contrapositive we have in Part (b). Contrapositive is equivalent to the original statements S, therefore this sentence is equivalent to the converse of S.

3. X is a set that contains both students and courses. **Zorn** is a student, and **Zukes** is a course. Predicate $S(x)$ means x is a student, and $C(x)$ means x is a course, $E(x, y)$ means x is enrolled in y, $EQ(x, y)$ means x equals y, and $P(x, y)$ means x is more popular than y. Use set X, **Zorn**, **Zukes**, the predicates above, and the logical connectives you have learned in our course to translate each sentence below, either from symbolic form to English, or from English to symbolic. In English, try to avoid symbols (e.g. x) and predicates (e.g. $C(x)$).

(a) One, and only one, student in X is more popular than **Zorn**.

Solution: "One and only one" means there exists one student x who is more popular than Zorn, and any student who is more popular than Zorn must be x. Symbolically,

$$\exists x \in X, S(x) \wedge P(x, \text{Zorn}) \wedge (\forall y \in X, (S(y) \wedge P(y, \text{Zorn})) \Rightarrow EQ(y, x))$$

(b) $\forall x \in X, (S(x) \wedge E(x, \textbf{Zukes})) \Rightarrow \neg P(x, \textbf{Zorn})$

Solution: All students who are enrolled in Zukes are not more popular than Zorn.

(c) $\exists x \in X, C(x) \wedge E(\text{Zorn}, x), \forall y \in X, (C(y) \wedge \neg EQ(y,x)) \Rightarrow P(y,x)$

Solution: Some course that Zorn is enrolled in is less popular than all other courses.

(d) The only course in X that **Zorn** is not enrolled in is **Zukes**

Solution:

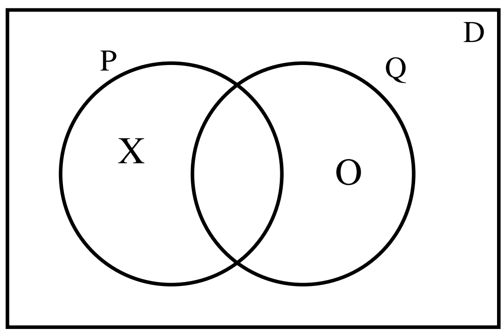
$$\neg E(\text{Zorn}, \text{Zukes}) \wedge (\forall x \in X, C(x) \wedge \neg E(\text{Zorn}, x) \Rightarrow EQ(x, \text{Zukes}))$$

4. For each pair of statements below, given an example of sets D, P, and Q that make one statement true and the other false. Explain the difference in words, and show it with a Venn diagram.

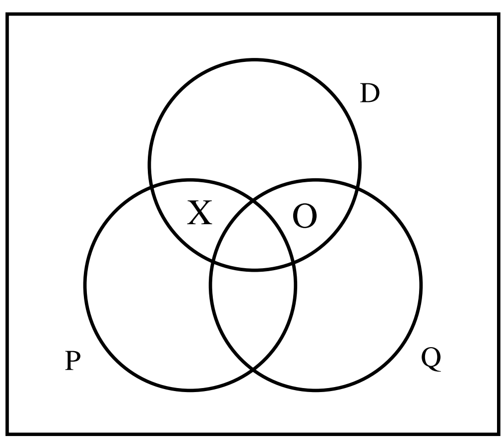
(a) The pair $\forall d \in D, P(d) \Rightarrow Q(d)$ and $\forall d \in D, P(d) \wedge Q(d)$.

Solution: We need to make $P \Rightarrow Q$ true and $P \wedge Q$ false, since $P \wedge Q$ being true would imply $P \Rightarrow Q$ being true.

Example 1: Let D be the universe of P and Q, an example Venn diagram looks like the following. The "X" sign makes sure $\forall d \in D, P(d) \Rightarrow Q(d)$ is true, and the "O" sign makes sure $\forall d \in D, P(d) \wedge Q(d)$ is false. A concrete example of this Venn diagram could be $P = \{1\}$, $Q = \{1, 2\}$ and $D = \{1, 2\}$.



Example 2: Let D be just another set like P and Q, an example Venn diagram looks like the following. The "X" sign makes sure $\forall d \in D, P(d) \Rightarrow Q(d)$ is true, and the "O" sign makes sure $\forall d \in D, P(d) \wedge Q(d)$ is false. A concrete example of this Venn diagram could be $P = \{1\}$, $Q = \{1, 2\}$ and $D = \{2\}$.



(b) The pair $\exists d \in D, P(d) \wedge Q(d)$ and $\exists d \in D, P(d) \Rightarrow Q(d)$.

Solution: Again, we need to make $P \Rightarrow Q$ true and $P \wedge Q$ false, since $P \wedge Q$ being true would imply $P \Rightarrow Q$ being true.

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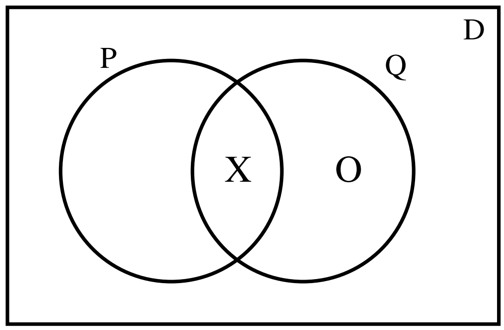

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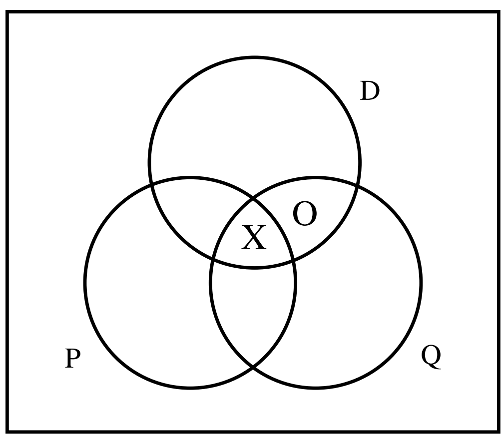

5. Suppose T is a set of natural numbers, and we have the following statement about T:

S2: Every element of T is 2 raised to some natural number power.

Which of the following statements imply $S2$? Which of the following statements are implied by $S2$? Explain.

(a) $T = \{64, 32, 128\}$.

Solution: This statement implies $S2$, since every element of T is a power of 2. However, $S2$ doesn't imply this statement, since there are plenty of sets consisting of powers of 2 that are different from $\{64, 32, 128\}$, such as $\{2, 4\}$

(b) If $i, j \in T$, and $i < j$, then i divides j.

Solution: $S2$ implies this statement, since if $S2$ is true, then smaller elements of T divide larger elements. However, this statement doesn't imply $S2$, since (again) $\{3, 6, 12\}$ satisfies this statement but not $S2$.

(c) T has no more than 1 odd member.

Solution: $S2$ implies this statement, since if $S2$ is true, then every member of T other than 1 is even. This statement doesn't imply $S2$, since $\{3, 6, 12\}$ has at most one odd member, but several of them are not powers of 2.

(d) No odd prime number divides any element of T.

Solution: $S2$ and this statement are equivalent, *i.e.,* this statement implies $S2$ and also is implied by $S2$. If every element of T is a power of 2, then the only prime factor of elements of T is 2, and conversely.

(e) T is the empty set.

Solution: This statement implies $S2$: every element of the empty set is a power of 2. However, $S2$ does not imply this statement, since $\{2\}$ satisfies $S2$ without being empty.