Symbolic Logic

1. Consider the following statements:

   (S1) Programs that passed test 1 also passed test 2.
   (S2) Programs passed test 2 unless they failed test 1.
   (S3) Programs passed test 2 only if they passed test 1.

   (a) Rewrite statements (S1), (S2), and (S3) using precise symbolic notation.

   Solution:
   Let P represent the set of all programs.
   Let $T_1(p)$ denote: "program p passed test 1".
   Let $T_2(p)$ denote: "program p passed test 2".
   Then, (S1) can be written: $\forall p \in P, T_1(p) \rightarrow T_2(p)$.
   (S2) can be written: $\forall p \in P, T_1(p) \leftrightarrow T_2(p)$.
   (S3) can be written: $\forall p \in P, T_2(p) \rightarrow T_1(p)$.

   Alternate Solution: Let $T(p,n)$ denote: "program p passed test n". Then, in the solution above, replace every "$T_1(p)$" with $T(p,1)$ and every "$T_2(p)$" with $T(p,2)$.

   (b) Which of the three statement have the same meaning?

   Solution: Based on the answers for part (a), none of the three statements have the same meaning (*i.e.*, all three have a different meaning).

2. Consider the statement:

   (S4) All Java programs passed test 1.

   (a) Rewrite (S4) using implication but no quantification.

   Solution: "If a program was written in Java, then it passed test 1."

   (b) Rewrite (S4) using precise symbolic notation.

   Solution:
   Let P represent the set of all programs, $J(p)$ denote: "program p was written in Java", and $T_1(p)$ denote: "program p passed test 1".
   Then, (S4) can be written: $\forall p \in P, J(p) \rightarrow T_1(p)$.

   Note: It would be reasonable but not as good to say instead:
   Let J represent the set of all Java programs and $T_1(p)$ denote: "program p passed test 1".
   Then, (S4) can be written $\forall p \in J, T_1(p)$.
   The reason why this is not as good as the first answer is because it makes it impossible to talk about programs other than the ones in J. If we wanted to talk about programs not written in Java, it would not be good enough (and it would be incorrect notation) to write something like "$\forall p \notin J$" because this would have the meaning of "for *everything* that is not a Java program" (including people, birds, colours, ...).
   The first answer is the best way to allow the possibility that every property mentionned in the statement could be true or false, while limiting the domain of discussion to the type of object we are interested in.

   (c) Write the contrapositive of (S4), symbolically and in English.

   Solution: $\forall p \in P, \neg T_1(p) \rightarrow \neg J(p)$.
   "Programs that failed test 1 were not written in Java."

   (d) Write the converse of (S4), symbolically and in English.

Solution: $\forall p \in P, T_1(p) \rightarrow J(p)$.
"All programs that passed test 1 were written in Java."

3. Draw a Venn diagram with sets to represent "programs written in C", "programs that passed test 1", and "programs that passed test 2" (make sure that your sets overlap to divide the diagram into eight regions). Then, for each program, write the program number in the appropriate region of your diagram (based on the information in the database above).

Solution:
Let P denote the set of all programs.
Let C denote the set of programs written in C.
Let T_1 denote the set of programs that passed test 1.
Let T_2 denote the set of programs that passed test 2.
The diagram is given in the figure below (or at the top of the next page).

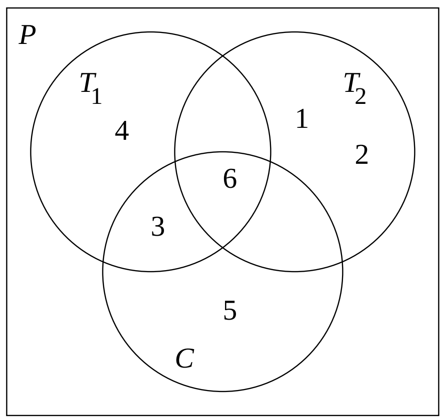


Figure 1: Venn diagram for Question 3.

4. Draw three copies of your diagram from the preceding question. On the first copy, shade the region(s) that corresponds to "programs that have passed every test". On the second copy, shade the region(s) that corresponds to "programs that have passed some text". On the third copy, share the region(s) that corresponds to "programs that have passed no test".

Solution: In the figures below or at the top of the next page, grey regions represent the "shaded" parts (white regions are "unshaded"). Also note that we take the meaning of "some" to be inclusive (*i.e.*, a program that passed every test is considered to have passed some test).

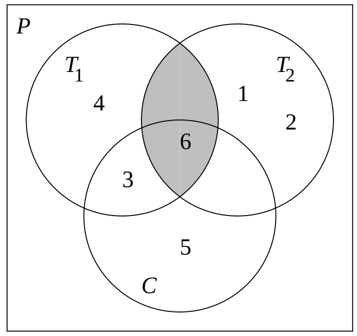


Figure 2: Programs that have passed every test.

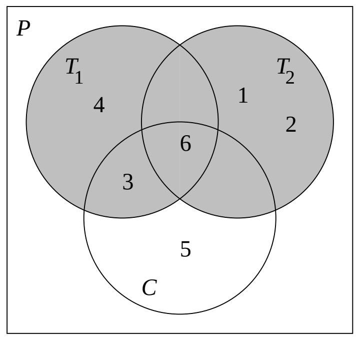


Figure 3: Programs that have passed some test.

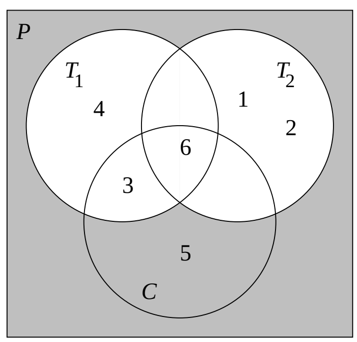


Figure 4: Programs that have passed no test.

5. State whether each statement below is true or false. When appropriate, justify your answer by citing a specific counter-example.

    (S5) Every Java program passed some test.
    (S6) Some Java program passed no test.
    (S7) No C program passed every test.

    Solution:

    (S5) is true, from Figure 3 (or directly from the database: program 1 passed test 2, program 2 passed test 2, and program 4 passed test 1).

    (S6) is false, from Figure 4 (also, because it is the negation of (S5)).

(S7) is false, from Figure 2: program 6 is a counter-example.

6. Let P denote the sentence "x is even". Let Q denote the sentence "x^2 is even". Write the sentence $P \rightarrow Q$ in English:

   (a) Using the words "if, then".

   (b) Using the word "implies".

   (c) Using the words "only if".

   (d) Using the words "is necessary for".

   (e) Using the words "is sufficient for".

   Solution:

   (a) If x is even, then x^2 is even.

   (b) x is even implies x^2 is even.

   (c) x is even only if x^2 is even.

   (d) It is necessary that x^2 be even for x to be even.

   Note that "x^2 is even is necessary for x is even" is not even a correct English sentence! Part of the point of this question was to make people realize that translating from symbolic notation to English involves more than just one-by-one replacement of symbols by words. You have to think about the meaning of the symbols and write a correct English sentence that has the same meaning.

   (e) It is sufficient that x be even for x^2 to be even.

7. Consider the following sentence about integers a, b, c:

   If a divides bc, then a divides b or a divides c.

   For each sentence below, state whether it is the same as, the negation of, the converse of, the contrapositive of, or unrelated to the statement above. Justify each of your answers briefly (*e.g.*, by writing both statements in symbolic notation).

   (a) If a divides b or a divides c, then a divides bc.

   (b) If a does not divide b or a does not divide c, then a does not divide bc.

   (c) a divides bc and a does not divide b and a does not divide c.

   (d) If a does not divide b and a does not divide c, then a does not divide bc.

   (e) a does not divide bc or a divides b or a divides c.

   (f) If a divides bc and a does not divide c, then a divides b.

   (g) If a divides bc or a does not divide b, then a divides c.

   Solution: The standard mathematical notation used to represent "divides" is a vertical bar, so for example, "a divides bc" can be represented symbolically as $a|bc$.

   It would also be perfectly fine to introduce a predicate letter like $D(x, y)$ to represent "x divides y" (so that "a divides bc" becomes $D(a, bc)$, for example), or even to introduce three unrelated letters to represent each of the three sentences (*e.g.*, P to represent "a divides bc", etc.), since we are only concerned with the logical structure of the various sentences, not their meaning.

   If we use the vertical bar notation, the original sentence can be written symbolically as follows:

$$a|bc \rightarrow a|b \vee a|c$$

(a) The sentence becomes: $a|b \vee a|c \rightarrow a|bc$.
This is the converse of the original sentence.

(b) The sentence becomes: $\neg(a|b) \vee \neg(a|c) \rightarrow \neg(a|bc)$.
This is unrelated to the original sentence.

(c) The sentence becomes: $a|bc \wedge \neg(a|b) \wedge \neg(a|c)$.
This is the negation of the original sentence.

(d) The sentence becomes: $\neg(a|b) \wedge \neg(a|c) \rightarrow \neg(a|bc)$.
This is the contrapositive of the original sentence.

(e) The sentence becomes: $\neg(a|bc) \vee a|b \vee a|c$.
This is equivalent to the original sentence.

(f) The sentence becomes: $a|bc \wedge \neg(a|c) \rightarrow a|b$.
This is equivalent to the original sentence.

(g) The sentence becomes: $a|bc \vee \neg(a|b) \rightarrow a|c$.
This is unrelated to the original sentence.

8. For each quantified statement below, rewrite the statement in English and state whether it is true or false. When appropriate, justify your answer with an example or counter-example.

(a) $\exists m \in \mathbb{N}\, \forall n \in \mathbb{N}, m > n$

(b) $\forall m \in \mathbb{N}\, \exists n \in \mathbb{N}, m > n$

(c) $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}, m > n$

Solution:

(a) One way to translate the statement is: "There exists a natural number larger than all other natural numbers."
This statement is clearly false: there is no "largest" natural number.

(b) One way to translate the statement is: "For all natural numbers, there is another natural number that is smaller."
This statement is false because there is no natural number smaller than 0 (so when $m = 0$, no value of n exists to satisfy $m > n$).

(c) One way to translate the statement is: "For all natural numbers, there is another natural number that is greater."
This statement is true because, for example, every natural number n has a successor $n + 1$.

9. In calculus, a function f with domain $\mathbb{R}$ (the real numbers) is defined to be *strictly increasing* provided that for all real numbers x and y, $f(x) < f(y)$ whenever $x < y$. Complete each of the following sentences using the appropriate symbolic notation.

(a) A function f with domain $\mathbb{R}$ is strictly increasing provided that ...

Solution: $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x < y \rightarrow f(x) < f(y)$

(b) A function f with domain $\mathbb{R}$ is not strictly increasing provided that ...

Solution: $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y \wedge f(x) \not< f(y)$

Alternate Solution: $\neg\big(\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x < y \rightarrow f(x) < f(y)\big)$

This is not as good as the first solution: in general, expressing the negation of a statement simply as "¬ statement" gives little information compared to taking the time to "work the negation in".

10. Find three sets A, B, C with as few elements as possible so that statement $(S1)$ below is true but statement $(S2)$ is false.

$$(S1) \quad \forall x \in A\ \exists y \in B, x + y \in C$$

$$(S2) \quad \exists y \in B\ \forall x \in A, x + y \in C$$

Solution: Let $A = \{0, 1\}$, $B = \{0, 1\}$, $C = \{1\}$.

Then, $(S1)$ is true because for $x = 0$, we can pick $y = 1$ and for $x = 1$, we can pick $y = 0$. However, $(S2)$ is false because when $y = 0$, $x = 0$ fails ($x + y \notin C$) and when $y = 1$, $x = 1$ fails.

11. At a murder trial, four witnesses give the following testimony.

Alice: If either Bob or Carol is innocent, then so am I.

Bob: Alice is guilty, and either Carol or Dan is guilty.

Carol: If Bob is innocent, then Dan is guilty.

Dan: If Bob is guilty, then Carol is innocent; however, Bob is innocent.

(a) Is the testimony consistent, *i.e.*, is it possible that everyone is telling the truth?

Solution: First, we define some abbreviations:

A: "Alice is innocent"

B: "Bob is innocent"

C: "Carol is innocent"

D: "Dan is innocent"

Then, we can rewrite the statements symbolically (with the understanding that "guilty = ¬ innocent"):

Alice: $(B \vee C) \rightarrow A$

Bob: $\neg A \wedge (\neg C \vee \neg D)$

Carol: $B \rightarrow \neg D$

Dan: $(\neg B \rightarrow C) \wedge B$

Now, let us assume that everyone's testimony is true. We will see that this leads to a contradiction, which means that someone must be lying.

Bob's testimony means that Alice is guilty. Then, the contrapositive of Alice's testimony means that both Bob and Carol are guilty. But this contradicts Dan's testimony that Bob is innocent.

(b) If every innocent (not guilty) person tells the truth and every guilty person lies, determine (if possible) who is guilty and who is innocent.

Solution: Using the same notation as above, there are many ways of determining who is innocent and who is guilty, including trial-and-error. One possibility is that Alice and Carol are guilty while Bob and Dan are innocent. Then, Alice's statement is false (because B is true but A is false), Bob's statement is true (because A is false and C is false), Carol's statement is false (because B is true and D is true), and Dan's statement is true (because B is true), as required.

12. Consider the following statement: (1)

"If a program has a syntax error, then the program will not compile."

(a) Define the domain and predicates necessary to translate the statement into precise symbolic notation.

Domain: The set of programs, P.

Let $S(x)$ represent x has a syntax error.

Let $C(x)$ represent x will compile.

(b) Translate (1) into precise symbolic notation. $\forall p \in P, S(p) \rightarrow \neg C(p)$

(c) (4 marks) Give the converse of (1) first in English, then in precise symbolic notation.

If a program does not compile, then it has a syntax error.

$\forall p \in P, \neg C(p) \rightarrow S(p)$

(d) Give the contrapositive of (1) first in English, then in precise symbolic notation.

If a program compiles then it does not have a syntax error.

$\forall p \in P, C(p) \rightarrow \neg S(p)$

(e) (4 marks) Give the contrapositive of your answer to 18c in precise symbolic notation.

$\forall p, \neg S(p) \rightarrow C(p)$ i.e., in words:

If a program does not have a syntax error then it will compile.

13. (40 marks) Assume you are given the following predicate symbols and your domain is $\mathbb{N}$, the set of natural numbers (we assume that $0 \in \mathbb{N}$).

$g(x, y)$: x *is greater than* y

$e(x, y)$: x *equals* y

$sum(x, y, z)$: $x + y = z$

$prod(x, y, z)$: $x \cdot y = z$

Translate the following statements into *idiomatic* English:

(a) $\forall x \in \mathbb{N}, g(x, 0)$

Every natural number is greater than 0.

(b) $\forall x \in \mathbb{N}, \exists z \in \mathbb{N}, prod(z, 2, x)$

Every natural number is even.

(c) $\forall a \in \mathbb{N}, \forall b \in \mathbb{N}, \forall c \in \mathbb{N}, (g(a, b) \wedge g(b, c)) \rightarrow g(a, c)$

For all natural numbers, if a is greater than b and b greater than c, then a is greater than c.

(d) $\neg(\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, g(m, n))$

There does not exist a largest number

We get this from first doing a literal translation;

It is not the case that, for any natural number there exists a greater number.

Translate the following English statements into precise symbolic notation. Only use the predicates and domain defined above. Make sure you specify the domain of your variables in your solution and that your predicates are boolean.

(e) Every positive multiple of 5 is greater than 7.

$\forall x \in \mathbb{N} \exists y \in \mathbb{N}, p(y, 5, x) \rightarrow g(x, 7)$

(f) There is an odd integer.

$\exists x \in \mathbb{N} \exists y \in \mathbb{N}, \exists z \in \mathbb{N}, prod(y, 2, z) \wedge sum(z, 1, x)$

(g) If $x + y = z$ then $y + x = z$.

$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, sum(x, y, z) \rightarrow sum(y, x, z)$

(h) Not all integers are multiples of 2.

$\neg \forall x \in \mathbb{N}, \exists y \in \mathbb{N}, prod(y, 2, x)$

14. Logic Puzzle: There are many brain teasers involving deserted islands and the people who inhabit them. One such puzzle, involves an island consisting of two different races. The Truth Tellers and the Liars. The Truth Tellers always tell the truth and the Liars, falsehoods. Suppose you meet three people U, V and W from this island. The first person U does not speak your language however V offers to translate. For each case, determine (if possible) from their responses to the following question, which race they each belong. If it is not possible, clearly show why it is not possible to determine which race at least one of U, V or W belong to.

How many of you are Truth Tellers?

Responses:

(a) V: "U said, '*Exactly one of us is a Truth Teller.*'"
W: "Don't believe V. He is a Liar".
If V is a Truth Teller then U must have said Exactly one of us is a Truth Teller. *Now if U is a liar, then there cannot be exactly one truth teller and W must be a truth teller–this contradicts V's statement. Therefore, U must be a truth teller which contradicts U's statement. Therefore, it is not possible to determine.*

(b) V: "U said, '*Exactly one of us is a Liar.*'"
W: "V's statement is true."
If V is telling the truth and U said Exactly one of us is a Liar *then both V and W are Truth Tellers. However, that means that U must be a Liar, which means that U told the truth and therefore cannot be a Liar. Therefore we know that V cannot have told the truth.*
Suppose that V is a Liar. Then U could not have said Exactly one of us is a Liar *and W's statement is a lie. Therefore W and V are liars. What about U? We don't know what U said, so we don't know whether U is a liar or not. Therefore we can not determine which race U belongs to.*

15. Consider our example from class about rainy days.

(2) *Every rainy day I bring an umbrella.*

(3) *If I bring an umbrella, then I stay dry.*

(a) For each of the following statements, determine whether it has the same meaning as (2). If it has a different meaning, make a small alteration to the statement so that it has the same meaning.
   i. I bring an umbrella, if it is a rainy day. *same*
   ii. If it is a rainy day, I bring an umbrella. *same*
   iii. I bring an umbrella only if it is a rainy day. *It is a rainy day only if I bring an umbrella.*
   iv. A rainy day is sufficient for me to bring an umbrella. *same*
   v. A rainy day is necessary for me to bring an umbrella. *For me to bring an umbrella it is necessary for it to be a rainy day.*

(b) Assume that it is a rainy day. What conclusions can you draw given statements (2) and (3). Explain your reasoning.
If it is a rainy day, then I willl bring an umbrella and since I have an umbrella, I will stay dry. Therefore I can conclude that I stay dry and have an umbrella.

(c) Now assume that I forgot my umbrella. What conclusions can you draw? Explain your reasoning.
If I don't have an umbrella, then it cannot be rainy since otherwise statement (2) would be false. However, I may or may not be dry because statement (3) is of the form $false \rightarrow p$ and is therefore true no matter what the truth value is of p.

16. Recall the table of hockey stats from class.

Number	Pos.	Player	Team	GP	G	A	PTS
1	C	Alexei Zhamnov	PHI	11	4	8	12
2	RW	Jarome Iginla	CAL	13	6	6	12
3	C	Joe Sakic	COL	11	7	5	12
4	C	Vincent Damphousse	SJ	11	5	6	11
5	RW	Martin St. Louis	TB	9	4	7	11
6	LW	Fredrik Modin	TB	9	5	6	11
7	C	Saku Koivu	MON	11	3	8	11
8	C	Peter Forsberg	COL	11	4	7	11
9	RW	Alexei Kovalev	MON	11	6	4	10

(a) Draw a Venn diagram with sets that show "players with 5 or more goals", "players who have played at least 11 games", "players who have more points than games played". Using the information in the table, enter each player's number into the appropriate region of the diagram.

(b) Make a copy of the diagram in (16a)and shade in the region that corresponds to the statement "All players who have played less than 11 games yet scored more points than games played".

(c) Make another copy of the diagram in (16a) and shade in the region that corresponds to the statement "Every player that has more points than games played and has scored at least 5 goals".

17. The following was heard on TV recently:

"Product X is so good, it won an award!"

The makers of product X want you to believe certain (possibly implicit) hypothesis/hypotheses and conclusion(s).

(a) Write these down explicitly.

Soln:

Hypotheses:

- Product X won an award.
- If a product wins an award then it is so good.

Conclusions: Product X is so good.

(b) Formalize them with appropriate domains and predicates.

Soln:

Let G(x) represent product x is so good. Let A(x) be product x won an award. $\forall p \in Products, A(p) \rightarrow P(p)$. Therefore, P(X), i.e., product X is so good.

18. There is a set P, of problems that are "polynomial time solvable" and a set NP, of problems that are only "non-deterministically polynomial time solvable". Consider the following statement:

(S1) *"If $P = NP$, then the problem SAT is polynomial time solvable."*

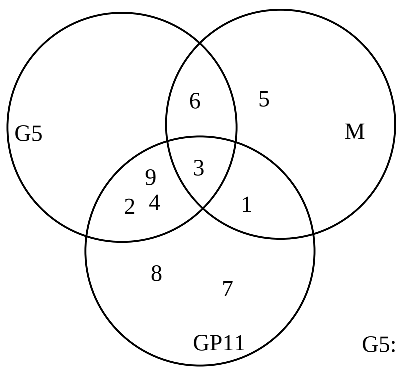
(a) Define the domain and predicates necessary to translate the statement into precise symbolic notation.

Soln:

We don't need a domain for this part. Let e(A,B) represent sets A and B are equal. Let p(x) represent that problem x is polynomial time solvable.

(b) Translate (S1) into precise symbolic notation. **Soln:**

$e(P, NP) \rightarrow p(SAT)$

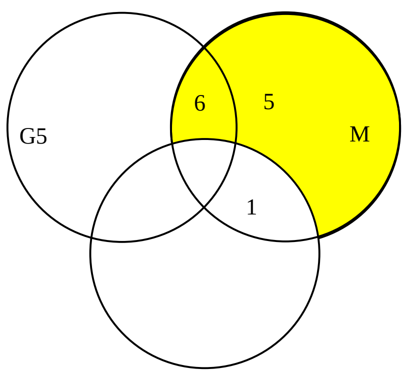


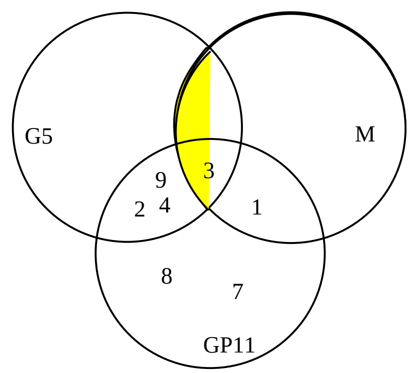
G5: players who have scored more than 5 goals

GP11: players who have played more than 11 games

M: players who have scored more points than games played

(a)





(b)

(c)

(c) Give the converse of (S1) first in English, then in precise symbolic notation.
Soln:
If SAT is polynomial time solvable then $P = NP$. $p(SAT)\ rightarrowe(P, NP)$

(d) Give the contrapositive of (S1) first in English, then in precise symbolic notation.
Soln:
If SAT is not polynomial time solvable then $P \neq NP$. $\neg p(SAT) \rightarrow \neg e(P, NP)$

(e) Give the contrapositive of your answer to 18c in precise symbolic notation.
Soln:
$\neg e(P, NP) \rightarrow \neg p(SAT)$

Now consider the same sentence expressed using quantification:

(S2) *If every problem is polynomial time solvable if and only if it is non-deterministically polynomial time solvable, then the problem SAT is polynomial time solvable."*

(f) Define the domain and predicates necessary to translate the statement into precise symbolic notation.
Soln:
Domain: X is set of all problems, p(x) represents x is polynomial time solvable, np(x) represents that x is non-deterministically polynomial time solvable.

(g) Translate (S2) into precise symbolic notation. Simplify your answer such that *only predicates* are negated (not entire sentences).
Soln:
$(\forall x \in X, p(x) \leftrightarrow np(x)) \rightarrow p(SAT)$

(h) Give the converse of (S2) first in English, then in precise symbolic notation. Simplify your answer such that *only predicates* are negated.
Soln:
If SAT is polynomial time solvable then every problem is polynomial time solvable if and only if it is non-deterministically polynomial time solvable.
Soln:
$\neg p(SAT) \rightarrow (\forall x \in X, p(x) \leftrightarrow np(x))$

(i) Give the contrapositive of (S2) first in English, then in precise symbolic notation. Simplify your answer such that *only predicates* are negated.
Soln:
If SAT is not polynomial time solvable then there exists a problem that is either in P but not in NP or not in P but in NP. $\neg p(SAT) \rightarrow \exists x \in X, [(\neg p(x) \wedge np(x)) \vee (p(x) \wedge \neg np(x))]$

(j) Give the contrapositive of your answer to 18h in precise symbolic notation. Simplify your answer such that *only predicates* are negated.
Soln:

$$\begin{aligned}
& \neg(\forall x \in X, p(x) \leftrightarrow np(x)) \rightarrow \neg p(SAT) \\
\Leftrightarrow\ & (\exists x \in X, \neg(p(x) \leftrightarrow np(x))) \rightarrow \neg p(SAT) \\
\Leftrightarrow\ & (\exists x \in X, \neg(p(x) \rightarrow np(x) \wedge np(x) \rightarrow p(x))) \rightarrow \neg p(SAT) \\
\Leftrightarrow\ & (\exists x \in X, \neg(p(x) \rightarrow np(x)) \vee \neg(np(x) \rightarrow p(x))) \rightarrow \neg p(SAT) \\
\Leftrightarrow\ & (\exists x \in X, \neg(\neg p(x) \vee np(x)) \vee (\neg(\neg np(x) \vee p(x)))) \rightarrow \neg p(SAT) \\
\Leftrightarrow\ & (\exists x \in X, (p(x) \wedge \neg np(x)) \vee (np(x) \wedge \neg p(x))) \rightarrow \neg p(SAT)
\end{aligned}$$

19. Consider the following database D of programs that test inputs. A program in this database may return a certificate when the input to the algorithm is accepted, rejected or both, and may or may not be linear in running time.

Program	Certificate	Linear
1	reject	yes
2	accept	no
3	accept	yes
4	reject	no
5	both	no
6	neither	yes

20. Now that we can write statements precisely, we can draw logical conclusions from a set of statements and *prove* that the conclusion is a consequence of the statements. These questions will help prepare you for learning to write proofs. For each set of statements, define a domain and set of predicates. Rewrite the statements and conclusions in precise symbolic notation. Assuming that the statements are true, determine which one of the possible conclusions can be drawn from the statements. Justify your choice of conclusion by explaining how the two statements imply the conclusion.

    Soln:
    Domain is P the set of people. Predicates are pow(x): x is a powerful person, pol(x): x is a politician, f(x): x is easily forgotten.

    (a) (S1) "*All politicians are powerful people.*"
    Soln:
    $\forall x \in P, pol(s) \rightarrow pow(x)$
    (S2) "*No powerful people are easily forgotten.*"
    Soln:
    $\neg\exists x \in P, pow(x) \wedge f(x)$ Possible conclusions:
    i. *People who are easily forgotten are politicians.* **Soln:**
    $\forall x \in P, f(x) \rightarrow pol(x)$
    ii. *Politicians are not easily forgotten.* **Soln:**
    $\forall x \in P, pol(x) \rightarrow \neg f(x)$
    iii. *No powerful people are politicians.* **Soln:**
    $\neg\exists x \in P, pow(x) \wedge pol(x)$
    iv. *Some easily forgotten people are politicians.* **Soln:**
    $\exists x \in P, f(x) \wedge pol(x)$
    v. *All politicians are easily forgotten.* **Soln:**
    $\forall x \in P, pol(x) \rightarrow f(x)$

    Conclusion ii). Since (S1) says that all politicians are powerful and (S2) says that powerful people are never forgotten, we can conclude that politicians are never forgotten.
    Soln:
    Domain: Let S be the set of characters from the Simpsons. Let t(x) represent x gets in trouble.

    (b) (S3) "*If Bart gets in trouble then either Homer or Milhouse get in trouble.*"
    Soln:
    $t(Bart) \rightarrow (t(Homer) \vee t(Milhouse))$
    (S4) "*Homer does not get in trouble.*
    Soln:
    $\neg t(Homer)$
    Possible conclusions:

i. *If Milhouse gets in trouble then Bart gets in trouble.* **Soln:**
$t(Milhouse) \rightarrow t(Bart)$

ii. *If Homer does not get in trouble then Milhouse does not get in trouble.* **Soln:**
$\neg t(Homer) \rightarrow \neg t(Milhouse)$

iii. *Milhouse gets in trouble if Bart gets in trouble.* **Soln:**
$t(Bart) \rightarrow t(Milhouse)$

iv. *Bart gets in trouble whenever Homer gets in trouble.* **Soln:**
$t(Homer) \rightarrow t(Bart)$

v. *Either Bart gets in trouble or Milhouse gets in trouble.* **Soln:**
$t(Bart) \vee t(Milhouse)$

Soln:
Conclusion? iii) Since either Homer or Milhouse get in trouble if Bart gets in trouble and we know Homer does **not** get in trouble, we can conclude that if Bart gets in trouble then Milhouse gets in trouble.

21. Assume you are given the following predicate symbols and your domain is $\mathbb{N}$, the set of natural numbers (we assume that $0 \in \mathbb{N}$).

$g(x, y)$: *x is greater than y*

$e(x, y)$: *x equals y*

$sum(x, y, z)$: $x + y = z$

$prod(x, y, z)$: $x \cdot y = z$

Each of the following statements is a mathematical property of the natural numbers. Translate the following statements into English and state the property.

(a) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, prod(x, y, x)$.

(b) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, sum(x, y, x)$.

(c) $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, (sum(x, y, z) \leftrightarrow sum(y, x, z))$

(d) $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, g(x, y) \wedge g(y, z) \rightarrow g(x, z)$

(e) $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, \forall t \in \mathbb{N}, [sum(x, y, z) \wedge prod(z, t, w)$
$\leftrightarrow (\exists u \in \mathbb{N}, \exists v \in \mathbb{N}, prod(x, t, u) \wedge prod(y, t, v) \wedge sum(u, v, w))]$

Now consider the following statements. Using the above predicates rewrite each statement in precise symbolic notation.

(f) x divides m. Recall that *divides* means that $xy = m$ for some number y. **Soln:**
$\exists c \in \mathbb{N}, product(x, c, m)$

(g) m is the smallest number that x divides. **Soln:**
$\exists c \in \mathbb{N}, product(x, c, m) \wedge \forall d \in \mathbb{N}, (\exists a \in \mathbb{N}, product(x, a, d)) \rightarrow g(d, c) \vee e(d, m)$

(h) $LCM(x, y, m)$: m is the smallest number that both x and y divide. **Soln:**
$\exists c \in \mathbb{N}, product(x, c, m) \wedge \exists d \in \mathbb{N}, prod(y, d, m) \wedge \forall a \in \mathbb{N}, \forall b \in \mathbb{N}, \forall z \in \mathbb{N}, prod(x, a, z) \wedge prod(y, b, z) \rightarrow L(m, z)$ [Hint: Do you need to quantify the variables x, y and m?]

(i) $GCD(a, b, c)$: a is the greatest common divisor of b and c. **Soln:**
$(\exists x \in \mathbb{N}, \exists y \in \mathbb{N}, prod(x, a, b) \wedge prod(y, a, c)) \wedge \forall d \in \mathbb{N}, \exists e \in \mathbb{N}, \exists f \in \mathbb{N}, prod(d, e, b) \wedge prod(d, f, c) \rightarrow L(d, a)$

22. Consider the following sentence:

(S1) *If* $m = 2^n - 1$ *is a prime number, then* n *is prime.*

[Note: these types of prime numbers are called *Mersenne Primes*.]

Rewrite (S1) without using "If . . ., then . . ." but using:

(a) "implies" **Soln:**
$m = 2^n - 1$ is a prime number implies that n is a prime.

(b) "is sufficient for" **Soln:**
$m = 2^n - 1$ being a prime number is sufficient for n to be a prime.

(c) "is necessary for" **Soln:**
n being prime is necessary for $m = 2^n - 1$ to be prime.

(d) "whenever" **Soln:**
n is prime whenever $m = 2^n - 1$ is a prime.

(e) "only if" **Soln:**
$m = 2^n - 1$ is a prime only if n is a prime.

(f) "requires" **Soln:**
$m = 2^n - 1$ is a prime requires that n is a prime.

23. Determine whether $\exists$ can be factored from an implication. In other words, is

$$\exists x \in X, (p(x) \rightarrow q(x)) \quad \Leftrightarrow \quad (\exists x \in X, p(x)) \rightarrow (\exists x \in X, q(x))$$

true? Explain your reasoning. Marks will only be given for your *explanation*.

Soln:
Consider when the domain is D={Bart, Homer} and $p(Bart) = T$ and $p(Homer) = F$, $q(Bart) = F$ and $q(Homer) = F$. Then the left side is satisfied by $p(Homer) \rightarrow q(Homer)$ but the right side cannot be satisfied since $p(Bart)$ satisfies the $\exists x \in D, p(x)$ but neither Homer nor Bart satisfy $q(x)$

24. For each set of sentences, define the domain X, the value of $a \in X$ (for part b), and the predicates $A(x)$ and $B(x)$ such that the last sentence is false and the other sentences are true.

(a)

$$\begin{array}{ll} (T) & \forall x \in X, A(x) \rightarrow B(x) \\ (F) & \exists x \in X, A(x) \wedge B(x) \end{array}$$

Soln:
Let X be $\{1\}$ and A(1)=F and B(1)=T

(b)

$$\begin{array}{ll} (T) & \forall x \in X, A(x) \rightarrow B(x) \\ (T) & \neg A(a) \\ (F) & \neg B(a) \end{array}$$

Soln:
Let X = $\{$a $\}$ and A(a) = F, B(a)=T

25. (10 marks) Let $p(n)$ and $q(n)$ represent the following predicates:

$$p(n) : n \text{ is odd} \quad q(n) : n^2 \text{ is odd}$$

where the domain is the set of integers. Determine which of the following statements are logically equivalent to each other.

Soln:
a,b,c,e,f,g are logically equivalent to each other. h,i are logically equivalent to each other.

(a) If the square of any integer is odd, then the integer is odd.
Soln:
$\forall n \in \mathbb{Z}, q(n) \rightarrow p(n)$

(b) $\forall n \in \mathbb{Z}, (p(n)$ is necessary for $q(n))$.
Soln:
$\forall n \in \mathbb{Z}, q(n) \rightarrow p(n)$

(c) The square of any odd integer is odd.
Soln:
$\forall n \in \mathbb{Z}, q(n) \rightarrow p(n)$

(d) There are some integers whose squares are odd.
Soln:
$\exists n \in \mathbb{Z}, q(n)$

(e) Given any integer whose square is odd, that integer is likewise odd.
Soln:
$\forall n \in \mathbb{Z}, q(n) \rightarrow p(n)$

(f) $\forall n \in \mathbb{Z}, \neg p(n) \rightarrow \neg q(n)$.
Soln:
$\forall n \in \mathbb{Z}, q(n) \rightarrow p(n)$

(g) Every integer with an odd square is odd.
Soln:
$\forall n \in \mathbb{Z}, q(n) \rightarrow p(n)$

(h) Every integer with an even square is even.
Soln:
$\forall n \in \mathbb{Z}, \neg q(n) \rightarrow \neg p(n)$

(i) $\forall n \in \mathbb{Z}, p(n)$ is sufficient for $q(n)$.
Soln:
$\forall n \in \mathbb{Z}, p(n) \rightarrow q(n)$

26. (a) Determine whether $\exists$ can be factored from an implication. In other words, is

$$\exists x \in X, (p(x) \rightarrow q(x)) \quad \Leftrightarrow \quad (\exists x \in X, p(x)) \rightarrow (\exists x \in X, q(x))$$

true?
Soln:
Consider $\rightarrow$ of the statement: i.e. $\exists x \in X, (p(x) \rightarrow q(x)) \rightarrow (\exists x \in X, p(x)) \rightarrow (\exists x \in X, q(x))$. Suppose we let the domain $X = \{x_1, x_2\}$, $p(x_1)$ = false, $p(x_2)$ = true and $q(x_1) = q(x_2)$ = false. Then the LHS is true but the RHS is false. Hence this is false.
And for $\leftarrow$: i.e. $\exists x \in X, (p(x) \rightarrow q(x)) \leftarrow (\exists x \in X, p(x)) \rightarrow (\exists x \in X, q(x))$. Suppose we now let the domain D be empty, p and q be any predicates. Then since there's no p, the RHS is true. But there's nothing in D, so there can't be an example for the LHS. Hence this is false also.
Combining the 2 results mean that the answer to this statement is false.

(b) Consider

$$\forall y \in D, \exists x \in D, p(x,y) \to \exists x \in D, \forall y \in D, p(x,y).$$

i. Define D and $p(x,y)$ such that the statement is true.
Soln:
Suppose we let the domain be 1 and let $p(x,y)$ be $x \leq y$. Then the left side is true and the right side is true. Therefore the statement is true.

ii. Define D and $p(x,y)$ such that the statement is false.
Soln:
Suppose we let the domain be $\mathbb{Z}$ and let $p(x,y)$ be $x < y$. Then the left side is true and the right side is false. Therefore the statement is false.

Note: there are other solutions.

(c) Consider the following statement:

$$(\forall x \in X, p(x) \to \neg\forall x \in X, q(x)) \leftrightarrow (\exists x \in X, \forall y \in X(p(x) \to \neg q(y)))$$

HINT: You may need to use the logical equivalence laws defined in class to simplify the statement first. If so, ***show and justify each step***.
Soln:
Consider the simplication of the LHS. We can say that

$$\begin{aligned}&\forall x \in X, p(x) \to \neg(\forall x \in X, q(x))\\ &\Leftrightarrow \neg(\forall x \in X, p(x)) \lor \neg(\forall y \in X, q(y))\\ &\Leftrightarrow \exists x \in X, \neg p(x) \lor \exists y \in X, \neg q(y)\end{aligned}$$

and for the RHS we notice that

$$\begin{aligned}&\exists x \in X, \forall y \in X, (p(x) \to \neg q(y))\\ &\Leftrightarrow \exists x \in X, \forall y \in X, (\neg p(x) \lor \neg q(y)\\ &\Leftrightarrow \exists x \in X, \neg p(x) \lor \forall y \in X, \neg q(y)\end{aligned}$$

i. Define D, $p(x)$ and $q(y)$ such that the statement is true.
Soln:
Suppose we let the domain $D = \{3\}$, let $p(x)$ = x is odd, and $q(x)$ = x is even. Then the LHS says, if every element of the domain, namely 3, is odd, then there exists some number (3) that is not odd, (since the only element in the domain is 3). The RHS says, if there exists a number that is odd (namely 3), then all numbers are not even (i.e. 3). Hence the statement is true.

ii. Define D, $p(x)$ and $q(y)$ such that the statement is false.
Soln:
Suppose we let the domain $D = \{3, 9\}$, let $p(x)$ = x is odd, and $q(x)$ = x is prime. Now the LHS says, if every element of the domain (3 and 9) is odd then there exists some element of the the domain that is not prime (9), which is true. The RHS says, if there exists an odd number in the domain (3 or 9), then all elements of the domain are not prime, which is false since 3 is a prime! Hence the statement is false. Note: there are other solutions.

27. (15 marks) Consider the statement

(1) *A number is rational if it can be written as $\frac{a}{b}$ where a and b are integers.*

(a) Define a domain and a set of predicates and rewrite the statement in precise symbolic notation. **Soln:**
Let the domain X be the set of all numbers. Let $D(a,b,c)$ = c is a number that can be written as $\frac{a}{b}$, $N(a)$= a is a natural number, $R(a)$ = a is a rational number. Symbolic notation: $\exists a \in X, \exists b \in X, \forall c \in$

$X, (N(a) \wedge N(b) \wedge D(a,b,c)) \rightarrow R(c)$.

Note: This is just one of many solutions.

(b) Rewrite (1) using

- "sufficient"
  Soln:
  It is sufficient that a number can be written as $\frac{a}{b}$ where a and b are integers.
- "only if"
  Soln:
  A number can be written as $\frac{a}{b}$ where a and b are integers only if that number is rational.
- "is necessary"
  Soln:
  It is necessary that a number is rational for it to be written as $\frac{a}{b}$ where a and b are integers.
- conjunction *instead* of implication.
  Soln:
  It is not the case that, a number can be written as $\frac{a}{b}$ where a and b are integers, and that number is not rational.

(c) Write the converse and the contrapositive of the converse of (1). Is the converse true? if so, how can we alter (1) to reflect this. If not, give a counter example.

Soln:
Converse of (1): If a number is rational then that number can be written as $\frac{a}{b}$ where a and b are integers.

Contrapositive of the converse (1): It is not the case that, if a number can be written as $\frac{a}{b}$ where a and b are integers, then that number isn't a rational.

Hence the converse is true. Thus we can alter (1) to:

A number is rational if and only if it can be written as $\frac{a}{b}$ where a and b are integers.

Sequences

1. Consider the following sentences about sequences, $a_0, a_1, a_2, \ldots$, of integers:

$$\begin{array}{ll} \text{(S1)} & \exists j \in \mathbb{N}, \forall i \in \mathbb{N}, i \neq j \rightarrow a_i \geq a_j \\ \text{(S2)} & \exists j \in \mathbb{N}, \forall i \in \mathbb{N}, i > j \rightarrow a_i < a_{i+1} \wedge i < j \rightarrow a_i \geq a_{i+1} \\ \text{(S3)} & \forall j \in \mathbb{N}, j > 0 \rightarrow (a_{2j} = -a_{4j} \wedge a_{2j+1} = -a_{2j+3}) \end{array}$$

(a) For each sentence describe in clear, precise English the property that a sequence must have to satisfy the sentence.

Soln:

(S1) There is a smallest element (or more than one).

(S2) The sequence is non-increasing until some element a_j and then increasing from a_j onwards.

(S3) The odd elements have the same magnitude and alternate in sign and the even elements are the negative of he element at twice the index.

(b) For each sentence, give the *negation* of the sentence.

Soln:

(S1)

$$\begin{aligned} & \neg\exists j \in \mathbb{N}, \forall i \in \mathbb{N}, i \neq j \rightarrow a_i \geq a_j \\ \Leftrightarrow\quad & \forall j \in \mathbb{N}, \exists i \in \mathbb{N}, \neg(\neg i \neq j \vee a_i \geq a_j) \\ \Leftrightarrow\quad & \forall j \in \mathbb{N}, \exists i \in \mathbb{N}, i \neq j \wedge a_i < a_j \end{aligned}$$

(S2)

$$\begin{aligned} & \neg\exists j \in \mathbb{N}, \forall i \in \mathbb{N}, i > j \rightarrow a_i < a_{i+1} \wedge i < j \rightarrow a_i \geq a_{i+1} \\ \Leftrightarrow\quad & \forall j \in \mathbb{N}, \exists i \in \mathbb{N}, \neg[(\neg i > j \vee a_i < a_{i+1}) \wedge (\neg i < j \vee a_i \geq a_{i+1})] \\ \Leftrightarrow\quad & \forall j \in \mathbb{N}, \exists i \in \mathbb{N}, \neg(\neg i > j \vee a_i < a_{i+1}) \vee \neg(\neg i < j \vee a_i \geq a_{i+1}) \\ \Leftrightarrow\quad & \forall j \in \mathbb{N}, \exists i \in \mathbb{N}, (i > j \wedge a_i \geq a_{i+1}) \vee (i < j \wedge a_i < a_{i+1}) \end{aligned}$$

(S3)

$$\begin{aligned} & \neg\forall j \in \mathbb{N}, j > 0 \rightarrow (a_{2j} = -a_{4j} \wedge a_{2j+1} = -a_{2j+3}) \\ \Leftrightarrow\quad & \exists j \in \mathbb{N}, \neg(\neg j > 0 \vee (a_{2j} = -a_{4j} \wedge a_{2j+1} = -a_{2j+3})) \\ \Leftrightarrow\quad & \exists j \in \mathbb{N}, j > 0 \wedge \neg(a_{2j} = -a_{4j} \wedge a_{2j+1} = -a_{2j+3}) \\ \Leftrightarrow\quad & \exists j \in \mathbb{N}, j > 0 \wedge (a_{2j} \neq -a_{4j} \vee a_{2j+1} \neq -a_{2j+3}) \end{aligned}$$

(c) For each of the following sequences, determine whether the sequence satisfies each of the (S1), (S2) and (S3). Justify your claim in English using an example or counter example whenever possible.

(A1) $10, 9, 8, 7, 9, 11, 13, 15, \ldots$

Soln:

(S1) Since there exists a $j = 3$ such that a_j is the minimum element, (A1) satisfies (S1)

(S2) Let $i = 3$, then the sequence is non-increasing (actually decreasing) until a_i and increasing thereafter.

(S3) Since there exists a $j = 2$ such that $a_{2j} \neq -a_{4j}$ $A1$ does not satisfy $S3$.

(A2) $-1, 2, 1, -2, -1, 2, 1, -2, \ldots$

Soln:

(S1) Since there exists a $j = 3$ such that a_j is the minimum element, (A2) satisfies (S1)

(S2) To show that (A2) does not satisfy $S2$, we consider the negation of $S2$. Notice that for any j if $a_j = -1$ then pick $i = j + 2$, so $i > j \wedge a_i \geq a_{i+1}$, if $a_j = 1$, pick $i = j + 3$ so $i > j \wedge a_i \geq a_{i+1}$. Similarly for $a_j = 2$ and $a_j = -2$.

(S3) Notice that this does not satisfy S3. We can see this by picking $j = 2$. Then $a_{2j} = -1$ and $a_{4j} = -1$. This satisfies the negation of (S3).

(A3) $1, 2, 4, 8, 16, 32, \ldots$

Soln:

(S1) Since there exists a $j = 0$ such that a_j is the minimum element, (A3) satisfies (S1)

(S2) To show that (A3) does satisfy $S2$, consider $j = 0$. Then the right side of $\wedge$ is always true and the left side is true because the sequence is increasing.

(S3) Since (A3) does not have negative numbers it immediately satisfies the negation of $(S3)$.

(d) Is there a sequence that satisfies all three of (S1), (S2) and (S3). If yes, give the sequence, if not, then explain why it is impossible.

Soln:

Not possible. To satisfy (S2) we need the sequence to increasing at some point. However, (S3) requires the sequence to alternate between positive and negative numbers. Therefore to satisfy (S3), there is never a value of j such that the sequence is increasing from a_j onwards.

2. (20 marks) Consider these sentences about sequence of integers $a_0, a_1, a_2, \ldots$.

(S1) $\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, (j > i) \rightarrow (a_j \geq 2a_i)$ where each $a_i \in \mathbb{N}$.

(S2) $\forall i \in \mathbb{N}, (i > 2) \rightarrow (a_i - a_{i-1} \leq a_{i-1} - a_{i-2})$ where each $a_i \in \mathbb{Z}$.

(S3) $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, (i < j \rightarrow a_i < a_{i+1}) \wedge (j < i \rightarrow a_i > a_{i+1})$ where each $a_i \in \mathbb{Z}$.

Consider these sequences and answer the questions below.

(A) -10 -9 -8 -5 -2 0 100 99 97 96 95 94 . . .

(B) 40 20 10 5 1 0 0 0 0 0 0 0 . . .

(C) 0 2 5 11 23 47 100 200 400 900. . .

(a) For each sentence describe the sequences that satisfy it.

Soln:

S(1) means: "every element in the sequence is at least twice as large as every previous elements in the sequence."

Soln:
S(2) means: ”for every three consecutive elements of the sequence, excluding the 1st element of the sequence, the difference between the first 2 of the 3 consecutive elements is greater or equal to the difference between the last 2 of this 3 consecutive elements.”

(b) For each sentence express it’s negation moving the negation as far in as possible.

Soln:
$\neg S(1) : \neg(\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, (j > i) \rightarrow (a_j \geq 2a_i))$
$\Leftrightarrow \exists i \in \mathbb{N}, \exists j \in \mathbb{N}, \neg((j > i) \rightarrow (a_j \geq 2a_i))$
$\Leftrightarrow \exists i \in \mathbb{N}, \exists j \in \mathbb{N}, \neg((j > i) \wedge \neg(a_j \geq 2a_i)$
$\Leftrightarrow \exists i \in \mathbb{N}, \exists j \in \mathbb{N}, (j \leq i) \wedge (a_j < 2a_i)$

Soln:
$\neg S(2) :$
$\neg(\forall i \in \mathbb{N}, (i > 2) \rightarrow (a_i - a_{i-1} \leq a_{i-1} - a_{i-2}))$
$\Leftrightarrow \exists i \in \mathbb{N}, \neg((i > 2) \rightarrow (a_i - a_{i-1} \leq a_{i-1} - a_{i-2}))$
$\Leftrightarrow \exists i \in \mathbb{N}, \neg(i > 2) \wedge \neg(a_i - a_{i-1} \leq a_{i-1} - a_{i-2})$
$\Leftrightarrow \exists i \in \mathbb{N}, (i \leq 2) \wedge (a_i - a_{i-1} > a_{i-1} - a_{i-2})$

(c) For each sentence determine whether the sentence is true for each of (A), (B) and (C). If it is true, explain why and if it is not true, show where the sequence fails.

Soln:
S(1) is false for (A), i=6 and j=7 is a counterexample.

S(1) is false for (B), i=2 and j=1 is a counterexample.

S(1) is true for (C), since every element is at least twice as large as a previous.

S(2) is false for (A), i=3 is a counterexample.

S(2) is false for (B), i=3 is a counterexample.

S(2) is false for (C), i=3 is a counterexample.

3. Consider the following statements about sequences of natural numbers $a_0, a_1, a_2, \ldots$:

$$(S3) \quad \forall i \in \mathbb{N}\, \exists j \in \mathbb{N}\, \forall k \in \mathbb{N}, k > j \rightarrow a_k \neq a_i$$

$$(S4) \quad \forall i \in \mathbb{N}\, \exists j \in \mathbb{N}, j > i \wedge (a_j > a_i \vee a_j < a_i)$$

$$(S5) \quad \forall i \in \mathbb{N} \Big(\big(\exists j \in \mathbb{N}, j > i \wedge a_j > a_i\big) \wedge \big(\exists j \in \mathbb{N}, j > i \wedge a_j < a_i\big) \Big)$$

And the following sequences:

$$(A1) \quad 1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 5, 6, \ldots$$

$$(A2) \quad 1, 2, 4, 8, 16, 32, 64, 128, 256, \ldots$$

$$(A3) \quad 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 1, 1, \ldots$$

For each sequence and each statement, state whether the statement is true or false for the sequence and justify briefly.

Solution: First, let us express the meaning of each statement in English.

The meaning of $(S3)$ is: "for every number a_i in the sequence, there is some point (j) after which all numbers are different from a_i" (*i.e.*, every number occurs a finite number of times in the sequence).

The meaning of $(S4)$ is: "for every number a_i in the sequence, there is some number later in the sequence that is either less than or greater than a_i".

The meaning of $(S5)$ is: "for every number a_i in the sequence, there is some later number that is greater than a_i and there is some later number that is smaller than a_i" (*i.e.*, every number in the sequence is eventually followed by a greater number and also by another smaller number).

$(S3)$ **for** $(A1)$**:** From the meaning above, $(S3)$ is true for $(A1)$ since every number occurs in $(A1)$ at most three times.

$(S3)$ **for** $(A2)$**:** From the meaning above, $(S3)$ is true for $(A2)$ since every number in $(A2)$ is unique.

$(S3)$ **for** $(A3)$**:** From the meaning above, $(S3)$ is false for $(A3)$ since the number 1 occurs infinitely often in $(A1)$.

$(S4)$ **for** $(A1)$**:** From the meaning above, $(S4)$ is true for $(A1)$ since every number in $(A1)$ is eventually followed by a greater number.

$(S4)$ **for** $(A2)$**:** From the meaning above, $(S4)$ is true for $(A2)$ since $(A2)$ is strictly increasing.

$(S4)$ **for** $(A3)$**:** From the meaning above, $(S4)$ is false for $(A3)$ since $(A3)$ ends with an infinite run of 1's.

$(S5)$ **for** $(A1)$**:** From the meaning above, $(S5)$ is false for $(A1)$ since there is no number less than 1 in the sequence (so $(S5)$ fails for $i = 0$).

$(S5)$ **for** $(A2)$**:** From the meaning above, $(S5)$ is false for $(A2)$ since there is no number less than 1 in the sequence (so $(S5)$ fails for $i = 0$).

$(S5)$ **for** $(A3)$**:** From the meaning above, $(S5)$ is false for $(A3)$ since there is no number less than 1 in the sequence (so $(S5)$ fails for $i = 9$).

4. (a) Proof that $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \rightarrow j < i$ for the sequence $\forall n \in \mathbb{N}, a_n = n^2$.

Let i = 2,
  Let $j \in \mathbb{N}$,
    Suppose $a_j \leq i = 2$
      then $a_j = j^2 \leq 2$
      So $j \leq \sqrt{2}$ and since $j \in \mathbb{N}$, $j \leq 1$
      So $j < 2 = i$
    $a_j \leq i \rightarrow j < i$
  Since $h \in \mathbb{N}$ and j arbitrary, $\forall j \in \mathbb{N}, a_j \leq i \rightarrow j < i$
Since $i = 2, i \in \mathbb{N}$, $\exists i \in \mathbb{N}, \forall j \mathbb{N}, a_j \leq i \rightarrow j < i$

(b) Disprove $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \rightarrow a_j = a_i$ for the sequence $\forall n \in \mathbb{N}, a_n = \lfloor n/2 \rfloor$.

First negate the sentence:

$$\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$$

Let $i \in \mathbb{N}$,
  Case 1: $i = 2k$ for some k,
    Let $j = i + 2$
      Then $j > i$ and $a_j = k$ and $a_i = k + 1$
      So $a_j \neq a_i$
    Since $j = i + 2 \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$
  Case 2: $i = 2k + 1$ for some $k \in \mathbb{N}$
    Let $j = i + 1$
      Then $j > i$ and $a_j = k$ and $a_i = 2(k + 1)/2 = k + 1$
      So $a_j \neq a_i$

Since $j = i + 1 \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i$
Since $i = 2k$ or $i = 2k + 1$ for some $k \in \mathbb{N}$,
$\exists j \in \mathbb{N}, j > i \land a_j \neq a_i$
Since $i \in \mathbb{N}, \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i$

(c) Proof that $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \neq a_i \rightarrow \exists k \in \mathbb{N}, k \neq a_k \land a_k > a_j$ for the sequence:

$$0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, \ldots$$

Let $i = 3$
Let $j \in \mathbb{N}$
Suppose $a_j \neq a_i$
Then let $k = j + 5$
Since $i = 3, a_i = 3$ and $a_j \neq 3$
Since $k = j + 5, k \geq 5$ and $k \neq a_k$ (since $a_k < 4$)
By definition of a_n and since $k = j + 5$, $a_k = 1$ if $a_j = 0$, $a_k = 2$ if $a_j = 1$ and $a_k = 3$ if $a_j = 2$
Therefore, $a_k > a_j$.
So $k \neq a_k \land a_k > a_j$
Since $k = j + 5 \in \mathbb{N}$, $\exists k \in \mathbb{N}, k \neq a_k \land a_k > a_j$
So $a_j \neq a_i \rightarrow \exists k \in \mathbb{N}, k \neq a_k \land a_k > a_j$
Since $j \in \mathbb{N}$
$\forall j \in \mathbb{N} a_j \neq a_i \rightarrow \exists k \in \mathbb{N}, k \neq a_k \land a_k > a_j$
Since $i = 3 \in \mathbb{N}$, $\exists i \in \mathbb{N}, \forall j \in \mathbb{N} a_j \neq a_i \rightarrow \exists k \in \mathbb{N}, k \neq a_k \land a_k > a_j$

General Proofs

1. Let $\mathbb{N}$ be the natural numbers $\{0, 1, 2, \ldots\}$, $\mathbb{Z}$ be the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and $\mathbb{R}$ be the real numbers. For $x \in \mathbb{R}$, define $r(x)$ as: $\exists m \in \mathbb{N}, \exists n \in \mathbb{N}, (n > 0) \wedge (x = m/n)$. You may assume $\neg r(\sqrt{2})$.

   Using our structured proof form, prove or disprove the following:

   (a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (r(x) \wedge r(y)) \Rightarrow r(x + y)$.

   Sample solution: The statement is true.

   Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$.
   Assume $r(x) \wedge r(y)$.
   Then $\exists m_x, n_x \in \mathbb{N}, n_x > 0 \wedge x = m_x/n_x$. (by definition of $r(x)$).
   Then $\exists m_y, n_y \in \mathbb{N}, n_y > 0 \wedge y = m_y/n_y$. (by definition of $r(y)$).
   Let $m_{x+y} = n_y m_x + n_x m_y$. Let $n_{x+y} = n_x n_y$.
   Then $m_{x+y} \in \mathbb{N}$. (since natural numbers are closed under multiplication and addition).
   Then $n_{x+y} \in \mathbb{N}$ and $n_{x+y} \neq 0$. (since natural numbers are closed under multiplication, and the product of non-zero natural numbers is not zero).
   Also $x + y = m_{x+y}/n_{x+y}$. (definition of addition of fractions).
   Hence $\exists m \in \mathbb{N}, \exists n \in \mathbb{N}\ n > 0$ and $x + y = m/n$.
   Thus $r(x + y)$. (by definition of $r(x + y)$).
   So $(r(x) \wedge r(y)) \Rightarrow r(x + y)$.
   Since x and y are arbitrary elements of $\mathbb{R}$, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (r(x) \wedge r(y)) \Rightarrow r(x + y)$.

   (b) The converse of (a)

   Sample solution: The statement is false.

   Let $x = \sqrt{2}$. Let $y = 2 - \sqrt{2}$.
   Then $x \in \mathbb{R}$. (real numbers include roots of positive reals).
   Then $y \in \mathbb{R}$. (real numbers are closed under subtraction).
   Let $m = 2$. Let $n = 1$.
   Then $m \in \mathbb{N}$.
   Then $n \in \mathbb{N}$ and $n \neq 0$.
   Also $x + y = 2/1$. (since $\sqrt{2} + 2 - \sqrt{2} = 2$).
   Hence $\exists m \in \mathbb{N}$ and $\exists n \in \mathbb{N}, n \neq 0$ and $x + y = m/n$.
   So $r(x + y)$ and $\neg r(x)$. (by definition of $r(x + y)$, and given assumption that $\neg r(\sqrt{2})$.
   Thus $r(x + y) \wedge \neg(r(x) \wedge r(y))$. (since $\neg r(x)$ implies $\neg r(x) \vee \neg r(y)$).
   Thus $r(x + y) \not\Rightarrow (r(x) \wedge r(y))$. (by negation of implication).
   Thus $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, r(x + y) \not\Rightarrow (r(x) \wedge r(y))$.

   (c) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (r(x) \wedge r(y)) \Rightarrow r(xy)$.

   Sample solution: The statement is true.

   Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$.
   Assume $r(x) \wedge r(y)$
   Then $\exists m_x, n_x \in \mathbb{N}, n_x > 0 \wedge x = m_x/n_x$. (by definition of $r(x)$).
   Then $\exists m_y, n_y \in \mathbb{N}, n_y > 0 \wedge y = m_y/n_y$. (by definition of $r(y)$).
   Let $m_{xy} = m_x m_y$. Let $n_{xy} = n_x n_y$
   Then $m_{xy} \in \mathbb{N}$. (natural numbers are closed under multiplication).
   Then $n_{xy} \in \mathbb{N}$ and $n_{xy} \neq 0$. (natural numbers are closed under multiplication and the product of non-zero natural numbers is non-zero).
   Also, $xy = m_{xy}/n_{xy}$.

Hence $\exists m \in \mathbb{N}, \exists n \in \mathbb{N}, n \neq 0$ and $xy = m/n$.
Thus $r(xy)$. (by definition of $r(xy)$).

So $(r(x) \wedge r(y)) \Rightarrow r(xy)$.

Since x and y are arbitrary elements of $\mathbb{R}$, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (r(x) \wedge r(y)) \Rightarrow r(xy)$.

(d) The converse of (c)

Sample solution: The statement is false

Let $x = \sqrt{2}$. Let $y = \sqrt{2}$.

Then $x \in \mathbb{R}$. (real numbers include roots of positive reals).
Then $y \in \mathbb{R}$. (real numbers include roots of positive reals).
Let $m = 2$. Let $n = 1$.

Then $m \in \mathbb{N}$.
Then $n \in \mathbb{N}$ and $n \neq 0$.
Also $xy = m/n$. (since $\sqrt{2}^2 = 2$).

Hence $\exists m \in \mathbb{N}, n \in \mathbb{N}, n \neq 0$ and $xy = m/n$.
So $r(xy)$ and $\neg r(x)$. (definition of $r(xy)$ and given assumption that $\neg r(\sqrt{2})$).
So $r(xy)$ and $\neg(r(x) \wedge r(y))$. (at least one of $r(x), r(y)$ is false).
So $r(xy) \not\Rightarrow (r(x) \wedge r(y))$. (negation of implication)

Thus $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, r(xy) \not\Rightarrow (r(x) \wedge r(y))$.

2. For $x \in \mathbb{R}$, define $|x|$ by

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Using our structured proof form, prove or disprove the following. You may assume that if $t > 0$, then $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow tx > ty$.

(a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x||y| = |xy|$.

Sample solution: The statement is true.

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$.

Case 1, $x < 0$ and $y < 0$.

Then $|x| = -x$ and $|y| = -y$. (definition of $|x|$ and $|y|$).
So $|x||y| = (-x)(-y) = xy$. (since $(-1)^2 = 1$).
Also $xy > 0$. (product of negative numbers is positive).
So $xy = |xy|$. (definition of $|xy|$ when $xy \geq 0$).
Hence $|x||y| = |xy|$.

Case 2, $x < 0$ and $y \geq 0$.

Then $|x| = -x$ and $|y| = y$. (definition of $|x|$ and $|y|$).
So $|x||y| = -xy$.
Also $xy \leq 0$. (product of a negative and a non-negative number is either 0 or negative).
So either $|xy| = -xy$ (by the definition of $|xy|$ when $xy < 0$), or $|xy| = xy = -xy$ (by the definition of $|xy|$ when $xy = 0$).
Thus $|x||y| = |xy|$.

Case 3, $x \geq 0$ and $y < 0$.

Then $|x| = x$ and $|y| = -y$. (definition of $|x|$ and $|y|$).
So $|x||y| = -xy$.
Also $xy \leq 0$. (product of a non-negative number with a negative number is non-positive).

So either $|xy| = -xy$ (by the definition of $|xy|$ for $xy < 0$) or $|xy| = xy = -xy$ (by the definition of $|xy|$ for $xy = 0$).
So $|x||y| = |xy|$.
Case 4, $x \geq 0$ and $y \geq 0$.
Then $|x| = x$ and $|y| = y$. (definition of $|x|$ and $|y|$).
Also $xy \geq 0$. (product of non-negative numbers is non-negative).
So $|xy| = xy$. (definition of $|xy|$).
Thus $|x||y| = |xy|$.
In each case $|x||y| = |xy|$, and these cover all possibilities. So $|x||y| = |xy|$.
Since x and y are arbitrary elements of $\mathbb{R}$, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x||y| = |xy|$.

(b) $\forall x_1 \in \mathbb{R}, \forall x_2 \in \mathbb{R}, \forall y_1 \in \mathbb{R}, \forall y_2 \in \mathbb{R}, |x_1| > |x_2| \wedge |y_1| > |y_2| \Rightarrow |x_1 y_1| > |x_2 y_2|$.

Sample solution: The statement is true.

Let $x_1 \in \mathbb{R}$. Let $x_2 \in \mathbb{R}$. Let $y_1 \in \mathbb{R}$. Let $y_2 \in \mathbb{R}$.
Assume $|x_1| > |x_2| \wedge |y_1| > |y_2|$.
Let $t = |y_1|$.
Then $t \in \mathbb{R}$. (definition of $|y_1|$).
Then $t > |y_2|$. (Since $t = |y_1| > |y_2|$, by assumption).
Also $|y_2| \geq 0$. (Since either $|y_2| = -y2 > 0$ if y_2 is negative, or $|y_2| = y_2 \geq 0$ if y_2 is non-negative, by definition of $|y_2|$.
So $t > 0$. (Since $t > |y - 2| \geq 0$).
Then $t|x_1| > t|x_2|$. (Since $|x_1| > |x_2|$, by assumption, and the result we are allowed to assume for this question).
So $|x_1||y_1| > |x_2||y_1|$. (by construction of t and commutativity of multiplication)
Let $t = |x_2|$.
Then $t \in \mathbb{R}$. (by definition of $|x_2|$.
Also $t \geq 0$. (since either $|x_2| = -x_2 > 0$ if x_2 is negative, or $|x_2| = x_2 \geq 0$ if x_2 is non-negative, by definition of $|x_2|$.).
So $t|y_1| \geq t|y_2|$. (since either $t|y_1| > t|y_2|$, by the result we are allowed to assume for this question when $t > 0$, or $t|y_1| = t|y_2|$ when $t = 0$).
So $|x_2||y_1| \geq |x_2||y_2|$. (by definition of t)
So $|x_1||y_1| > |x_2||y_2|$. (Since $|x_1||y_1| > |x_2||y_1|$ and $|x_2||y_1| \geq |x_2||y_2|$.
So $|x_1 y_1| > |x_2 y_2|$. (Since $|x_1||y_1| = |x_1 y_1|$ and $|x_2||y_2| = |x_2 y_2|$, by part (a)).
So $|x_1| > |x_2| \wedge |y_1| > |y_2| \Rightarrow |x_1 y_1| > |x_2 y_2|$.
Since x_1, x_2, y_1, y_2 are arbitrary elements of $\mathbb{R}$, $\forall x_1 \in \mathbb{R}, \forall y_1 \in \mathbb{R}, \forall y_2 \in \mathbb{R}, (|x_1| > |x_2| \wedge |y_1| > |y_2|) \Rightarrow |x_1 y_1| > |x_2 y_2|$.

3. Let $\mathbb{R}^+$ be the set of positive real numbers. Use our structured proof form to prove or disprove:

(a)
$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

Sample solution: The statement is (strangely) true.

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$.
Case 1, $x = y$
Let $\delta = 17$.
Then $\delta \in \mathbb{R}^+$
Let $\epsilon \in \mathbb{R}^+$.
Then $|x^2 - y^2| = 0 < \epsilon$. (since $x = y$ and ϵ is positive).
So $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$. (since the consequent is true, the entire implication is true).

Since ϵ is an arbitrary element of $\mathbb{R}^+$, $\forall \epsilon \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

So $\exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

Case 2, $x \neq y$.

Then $|x-y| > 0$. (since $x-y \neq 0$ and either $|x-y| = x-y$, if $x > y$, or $|x-y| = y-x$, if $y > x$).

Let $\delta = |x - y|/2$.

Then $\delta \in \mathbb{R}^+$. (since δ is half of a positive number).

Also, $|x - y| > \delta$. (since $|x - y| - \delta = \delta > 0$).

Let $\epsilon \in \mathbb{R}$.

Then $\neg(|x - y| < \delta)$. (since $|x - y| > \delta$).

So $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$. (since a false antecedent implies anything).

Since ϵ is an arbitrary element of $\mathbb{R}^+$, $\forall \epsilon \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

So $\exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

In either case $\exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$, and these cases cover all possibilities.

So $\exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

Since x and y are arbitrary elements of $\mathbb{R}$, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists \delta \in \mathbb{R}^+, \forall \epsilon \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

(b)

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

Sample solution: The claim is true.

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$. Let $\epsilon \in \mathbb{R}$.

Case 1, $x^2 - y^2 = 0$.

Let $\delta = 1$.

Then $\delta \in \mathbb{R}^+$.

Assume $|x - y| < \delta$.

Then $|x^2 - y^2| < \epsilon$. (since $\epsilon \in \mathbb{R}^+$)

So $|x - y| < \delta| \Rightarrow |x^2 - y^2| < \epsilon$.

So $\exists \delta \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$

Case 2, $x^2 - y^2 \neq 0$

Then $(x - y)(x + y) \neq 0$. (by factoring quadratic).

So $(x - y) \neq 0$ and $(x + y) \neq 0$. (non-zero product has no zero factors).

Let $\delta = \epsilon/(2|x + y|)$.

Then $\delta \in \mathbb{R}^+$. (since it ratio of positive numbers).

Assume $|x - y| < \delta$.

Then $|x^2 - y^2| = |x - y||x + y|$. (by part 2a).

So $|x^2 - y^2| < \delta|x + y|$. (by assumption that $|x - y| < \delta$).

So $\delta|x + y| = [\epsilon/(2|x + y|)] \times |x + y|$. (by construction of δ).

So $|x^2 - y^2| < \epsilon/2 < \epsilon$. (Since $\epsilon - \epsilon/2 = \epsilon/2 > 0$).

So $|x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

So $\exists \delta \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

In either case $\exists \delta \in \mathbb{R}^+, |x - y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$, and this covers all possibilities.

Since x and y are arbitrary elements of $\mathbb{R}$, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, |x-y| < \delta \Rightarrow |x^2-y^2| < \epsilon$

(c)

$$\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in [-1,1], \forall y \in [-1,1], |x-y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

Sample solution: The claim is true.

Let $\epsilon \in \mathbb{R}^+$.

Let $\delta = \epsilon/2$.

Then $\delta \in \mathbb{R}^+$. (since $\epsilon \in \mathbb{R}^+$ means $\epsilon/2$ is a ratio of positive real numbers).
Let $x \in [-1,1]$. Let $y \in [-1,1]$.
Assume $|x-y| < \delta$.

Then $(x+y) \in [-2,2]$. (Taking the maximum and minimum sums).
So $|x+y| \leq 2$. (Taking the maximum absolute value).
So $|x-y||x+y| \leq |x-y|2$.
So $|x^2 - y^2| \leq |x-y|2 < 2\delta$. (By the assumption that $|x-y| < \delta$).
So $|x^2 - y^2| < 2\epsilon/2 = \epsilon$. (By the construction of ϵ).

So $|x-y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.
Since x and y are arbitrary elements of $[-1,1]$, $\forall x \in [-1,1], \forall y \in [-1,1], |x-y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

So $\exists \delta \in \mathbb{R}^+, \forall x \in [-1,1], \forall y \in [-1,1], |x-y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

Since ϵ is an arbitrary element of $\mathbb{R}$, $\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in [-1,1], \forall y \in [-1,1], |x-y| < \delta \Rightarrow |x^2 - y^2| < \epsilon$.

(d)

$$\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x-y| < \delta \Rightarrow |x^2 - y^2| < \epsilon.$$

Sample solution: The claim is false, so I will prove its negation:

$$\exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x-y| < \delta \wedge |x^2 - y^2| \geq \epsilon.$$

Let $\epsilon = 1$.

Let $\delta \in \mathbb{R}^+$.

Let $x = 2/\delta$. Let $y = x + \delta/2$.

So $|x-y| = \delta/2 < \delta$. (since $\delta - \delta/2 = \delta/2 > 0$).
So $|x^2 - y^2| = |x-y||x+y| = \delta/2(2 \times 2/\delta + \delta/2)$ (by construction of x and y).
So $|x^2 - y^2| = 2 + \delta^2/4 > \epsilon$. (Since $\epsilon = 1$, and $\delta^2/4$ is positive).
So $|x-y| < \delta$ and $|x^2 - y^2| > \epsilon$.

So $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x-y| < \delta \wedge |x^2 - y^2| \geq \epsilon$.

Since δ is an arbitrary element of $\mathbb{R}^+$, $\forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x-y| < \delta \wedge |x^2 - y^2| \geq \epsilon$

So $\exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x-y| < \delta \wedge |x^2 - y^2| \geq \epsilon$

4. Suppose f and g are functions from $\mathbb{R}$ onto $\mathbb{R}$. Consider the following statements:

S1 $\quad \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (f(x) = f(y)) \Rightarrow (x = y).$
S2 $\quad \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (g(x) = g(y)) \Rightarrow (x = y).$
S3 $\quad \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (g(f(x)) = g(f(y))) \Rightarrow (x = y).$

Does (S1$\wedge$S2) imply S3? Prove your claim.

Sample solution: The claim is true.

Assume S1 $\wedge$ S2

So S1

So S2

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$.

Assume $g(f(x)) = g(f(y))$.

Let $x' = f(x)$. Let $y' = f(y)$.

Then $x' \in \mathbb{R}$ and $y' \in \mathbb{R}$. (by assumption that f and g are from $\mathbb{R}$ to $\mathbb{R}$).

So $x' = y'$. (By assumption of S2, since $g(x') = g(y')$).

So $f(x) = f(y)$. (by construction of x' and y').

So $x = y$. (By assumption of S1, since $f(x) = f(y)$).

So $g(f(x)) = g(f(y)) \Rightarrow x = y$.

Since x and y are arbitrary elements of $\mathbb{R}$, $g(f(x)) = g(f(y)) \Rightarrow x = y$.

Then S3. (by definition of S3).

Hence S1 $\wedge$ S2 $\Rightarrow$ S3.

Loop Invariants

1. Consider the following algorithm to compute the factorial of a number n.

   int FACTORIAL(int n)
   int i = 1;
   int f = 1;
   while ($i \leq n$)
   $f \leftarrow f \times i$;
   $i \leftarrow i + 1$;
   end while
   return (f);
   end FACTORIAL

   (a) We will denote the values of i and f on the k^{th} iteration of the loop by i_k and f_k. What are the values of i_k, f_k, i_{k+1}, f_{k+1}.
   Soln.
   $i_k = k + 1$, $f_k = k!$, $i_{k+1} = k + 2$, $f_{k+1} = (k + 1)!$

   (b) Fill in the blanks below to define a loop invariant for FACTORIAL. Note that k represents the iteration number.

   $$(LI)\ \ \forall k \in \mathbb{N}, (i_k = \_\_\_\_\_\_\_\_ \wedge f_k = \_\_\_\_\_\_\_\_ \wedge i_k \leq n) \rightarrow (i_{k+1} = \_\_\_\_\_\_\_\_ \wedge f_{k+1} = \_\_\_\_\_\_\_\_)$$

   Soln.

   $$(LI)\ \ \forall k \in \mathbb{N}, (i_k = k + 1 \wedge f_k = k! \wedge i_k \leq n) \rightarrow (i_{k+1} = k + 2 \wedge f_{k+1} = (k + 1)!)$$

   (c) Write a carefully structured proof that the loop invariant (LI) is true.
   Soln.
   Let $k \in N$
   Suppose that ($i_k = k + 1 \wedge f_k = k! \wedge i_k \leq n$) is true
   Then when we execute the code in the loop, $f \leftarrow f \times i$, $i \leftarrow i + 1$.
   Therefore $f_{k+1} = f_k i_k = k!(k + 1) = (k + 1)!$ and $i_{k+1} = i_k + 1 = k + 2$.
   Therefore $(i_k = k + 1 \wedge f_k = k! \wedge i_k \leq n) \rightarrow (i_{k+1} = k + 2 \wedge f_{k+1} = (k + 1)!)$.
   Since k was an arbitrary element of $\mathbb{N}$, therefore
   $\forall k \in \mathbb{N}, (i_k = k + 1 \wedge f_k = k! \wedge i_k \leq n) \rightarrow (i_{k+1} = k + 2 \wedge f_{k+1} = (k + 1)!)$.
   (15 marks - if did not use factorial in invariant, than max possible is 12)

2. Write a Java method `addBinary`, which takes two 1D `boolean` arrays of length n, called A and B.
   A and B represent two n-digit binary numbers, where:
   n = max{minimum number of digits needed to represent the first binary number,
   minimum number of digits needed to represent second the binary number}.

   `addBinary` adds the two numbers together, and returns the result in a `boolean` array (the length of the array should be equal to the minimum number of digits needed to represent the result) .

   If one of the numbers has fewer digits than the other, then you may assume that it is padded with 0s (i.e., there are `falses` in front of the first `true`). For example, you may assume that the arrays used to add 1101 and 10 would be:

```
boolean[] A = {true, true, false true};
boolean[] B = {false, false, true, false};
```

Comment your code. Do not import any Java packages. You may assume that A and B are not `null`, and that A and B are both positive (non-zero) numbers.

Solution:

```
public boolean[] addBinary(boolean[] A, boolean[] B) {
    boolean[] result = new boolean[A.length + 1];
    boolean C = false;
    boolean flag = false;

    for(int i = A.length - 1; i >= 0 ; i--) {
      if ((A[i] && B[i] && !C) || (A[i] && !B[i] && C) || (!A[i] && B[i] && C)) {
        C = true;
        result[i+1] = false;
        flag = false;
      } else if (A[i] || B[i] || C){
        C = false;
        result[i+1] = true;
        flag = true;
      } else {
        C = false;
        result[i+1] = false;
        flag = true;
      }
    }
    result[0] = C;

    boolean temp[] = new boolean[A.length];

    // If n+1st digit is not necessary, remove it.
    if(flag) {
      for(int i = 0; i < A.length; i++) {
        temp[i] = result[i+1];
      }
      return temp;
    } else {
      return result;
    }
  }
```

3. Let `A` be a 1D array of n natural numbers, and consider the following pseudocode, with the precondition, postcondition and loop invariants inserted as comments:

//pre-condition: $\forall k \in \mathbb{N}, 0 \leq k < n \rightarrow 0 \leq \mathtt{A}[k]$
$j = 0$
// *Outer loop invariant:* $\forall s \in \mathbb{N}, 0 < j \leq s < n \rightarrow 0 \leq \mathtt{A}[0] \leq \mathtt{A}[1] \leq ... \leq \mathtt{A}[j-1] \leq \mathtt{A}[s]$
// Outer loop description: *the first j elements are sorted in ascending order,*
// and the jth element is less than or equal to any element between the j+1st and the nth element.
while$(j < n)\{$
  $\mathtt{imin} = j$
  $k = j + 1$

```
// Inner loop invariant: ∀r ∈ ℕ, j ≤ r < k → A[imin] ≤ A[r]
// Inner loop desciption: finding smallest element between the j+1st and the nth element.
while(k < n){
   if(A[k] < A[imin]){
      imin = k
   }
   k++
} //end of inner loop
temp = A[j]
A[j] = A[imin]
A[imin] = temp
j++
} //end of outer loop
//post-condition: ∀j ∈ ℕ, ∀k ∈ ℕ, 0 ≤ j < k < n → A[j] ≤ A[k]
```

(a) Prove that the outer loop invariant is true at the start of the first iteration of the outer loop. **Soln:**
From the first assignment statement before the loop, we know $j = 0$. Since $j = 0$, the outer loop invariant is vacuously true.

(b) Prove that if the outer loop invariant is true at the start of an iteration of the outer loop and the outer loop is executed, then the inner loop invariant is true at the start of the first iteration of the inner loop.
Soln:
Suppose the outer loop invariant is true:
$\forall s \in \mathbb{N}, j \leq s \leq n \rightarrow 0 \leq \mathtt{A}[0] \leq ...\mathtt{A}[j-1] \leq \mathtt{A}[s]$

Since there is an iteration of the outer loop, $j < n$.
From the body of the outer loop, $imin = j$ and $k = j + 1$.
The inner loop invariant becomes: $\forall r \in \mathbb{N}, j \leq r < j + 1 \rightarrow \mathtt{A}[j] \leq \mathtt{A}[r]$

Let $r \in \mathbb{N}$
  Suppose $j \leq r < j + 1$.
    So $r = j$
    Thus $\mathtt{A}[j] \leq \mathtt{A}[r]$
  Therefore, $j \leq r < j + 1 \rightarrow \mathtt{A}[j] \leq \mathtt{A}[r]$
Since r is an arbitrary natural number, $\forall r \in \mathbb{N}, j \leq r < j + 1 \rightarrow \mathtt{A}[j] \leq \mathtt{A}[r]$

(c) Prove that if the inner loop invariant is true at the start of *any* iteration of the inner loop, then it is true at the end of that iteration. **Soln:**
Let $j', k', r', \mathtt{imin}'$ be the values of $j, k, r, \mathtt{imin}$ at the start of an iteration of the inner loop, and $j'', k'', r'', \mathtt{imin}''$ be the values at the end of the iteration.

Suppose the inner loop invariant is true at the start of an iteration of the inner loop:
$\forall r \in \mathbb{N}, j' \leq r < k' \rightarrow \mathtt{A}[\mathtt{imin}] \leq \mathtt{A}[r]$

Let $r \in \mathbb{N}$
  Suppose $j'' \leq r < k''$
    $k'' = k' + 1$ (by the last line of the inner loop)
    $j'' = j'$ (since j' is unchanged in the inner loop)
    Thus $j' \leq r < (k' + 1)$, so $j \leq r \leq k'$

Case 1: $j \le r < k'$
Since $\forall r \in \mathbb{N}, j' \le r < k' \rightarrow \texttt{A}[\texttt{imin}'] \le \texttt{A}[r]$ and $j' \le r < k'$, then $\texttt{A}[\texttt{imin}'] \le \texttt{A}[r]$
Case 1a: $\texttt{A}[k'] < \texttt{A}[\texttt{imin}']$
So $\texttt{imin}'' = k'$
Thus $\texttt{A}[imin''] \le \texttt{A}[r]$ (since $\texttt{A}[k'] < \texttt{A}[\texttt{imin}']$ and $\texttt{A}[imin'] \le \texttt{A}[r]$)
Case 1b: $\texttt{A}[k'] \ge \texttt{A}[\texttt{imin}']$
So $\texttt{imin}'' = \texttt{imin}'$
Thus $\texttt{A}[imin''] \le \texttt{A}[r]$ (since $\texttt{A}[imin'] \le \texttt{A}[r]$)
Case 2: $j \le r \wedge r = k'$
Case 2a: $\texttt{A}[k'] < \texttt{A}[\texttt{imin}']$
So $\texttt{imin}'' = k'$, thus $imin'' = r$ (since $r = k'$)
Therefore, $\texttt{A}[imin''] \le \texttt{A}[r]$
Case 2b: $\texttt{A}[k'] \ge \texttt{A}[\texttt{imin}']$
Therefore, $\texttt{A}[imin''] \le \texttt{A}[r]$, (since $k' = r$)
In all cases, $\texttt{A}[imin''] \le \texttt{A}[r]$
Therefore, $j'' \le r < k'' \rightarrow \texttt{A}[imin''] \le \texttt{A}[r]$
Since r is an arbitrary natural number, $\forall r \in \mathbb{N}, j'' \le r < k'' \rightarrow \texttt{A}[\texttt{imin}''] \le \texttt{A}[r]$

(d) Prove that if the outer loop invariant is true when the outer loop terminates, then the post-condition is true.
Soln:
When the outer loop terminates $j = n$ and the postcondition is vacuously true.

4. Consider the following algorithm (the lines starting with "//" are comments that will not be executed).

// precondition: $a \in \mathbb{N}$, $b \in \mathbb{N}$
m := a
// loop invariant: $m \ge 0 \wedge \exists x \in \mathbb{N}, a = bx + m$
while $m \ge b$ **do**
    m := $m - b$
end while
// postcondition: $m = a \bmod b$, *i.e.*, $0 \le m < b \wedge \exists x \in \mathbb{N}, a = bx + m$.

(a) Prove that the loop invariant is true before the first iteration of the loop.

SOLN:
Before the loop starts, $m = a$ so $m \ge 0$ (since $a \in \mathbb{N}$ by the precondition). Also, $a = m = 0b + m$ so $\exists x \in \mathbb{N}, a = bx + m$.

(b) Let m' denote the value of m at the end of some iteration of the loop and assume that the loop invariant is true for m'. Furthermore, let x' denote the value that makes $a = bx' + m'$ true.
Prove that the loop invariant remains true at the end of the next iteration of the loop, if there is such an iteration (use m'' to denote the value of m at the end of the next iteration).

SOLN:
Suppose that $m' \ge 0 \wedge a = bx' + m'$, and that the loop body executes once more.

Then, $m'' = m' - b$ (from the loop body), and since $m' \ge b$ (from the loop condition), $m'' = m' - b \ge 0$.
Let $x'' = x' + 1$. Since $m'' = m' - b$ iff $m'' + b = m'$,

$$a = bx' + m' = bx' + (m'' + b) = b(x' + 1) + m'' = bx'' + m''$$

so $\exists x \in \mathbb{N}, a = bx + m''$.
Hence, $m'' \ge 0 \wedge \exists x \in \mathbb{N}, a = bx + m''$.

(c) Prove that the postcondition is true after the last iteration of the loop. (You may assume that the loop invariant is true at the end of every iteration of the loop.)

SOLN:
At the end of the loop, we know that $m \geq 0 \wedge \exists x \in \mathbb{N}, a = bx + m$ (from the loop invariant). Moveover, the loop will stop when the loop condition becomes false, which allows us to conclude $m < b$ (from the negation of the loop condition). Hence, $0 \leq m < b \wedge \exists x \in \mathbb{N}, x = bx + m$.

5. **Sample solution:** The statement is false. First I put the statement into precise form, and then prove its negation.

The statement to be disproved is: $\forall x', y', z', x'', y'', z'', m, n \in \mathbb{Z}$, if $z' = mn - x'y'$, and (x', y', z') are the values of (x, y, z) before the loop iteration and (x'', y'', z'') are the values of (x, y, z) after the loop iteration, then $z'' = mn - x''y''$.

The negation of this statement is: $\exists x', y', z', x'', y'', z'', m, n \in \mathbb{Z}$, $z' = mn - x'y' \wedge (x', y', z')$ are the values of (x, y, z) before the loop iteration, (x'', y'', z'') are the values of (x, y, z) after the loop iteration $\wedge$ $z'' \neq mn - x''y''$. I now prove the negation.

Let $x' = m = 3$, $n = y' = 2$, $z' = 0$, $x'' = -2$, $y'' = 4$, and $z'' = 2$, and assume (x', y', z') are the values of (x, y, z) before the loop iteration.

Then $x, y, z, x', y', z', m, n$ are integers, and $mn - xy = 6 - 6 = 0 = z'$.
Since $x \neq 0$ and $x = 3$ is odd and $x * m = 9 > 0$, the program sets the new value of z to $0 - y = -2$, so $z'' = -2$ is the value of z after this iteration.
The program sets x'' to $\lfloor 3/(-2) \rfloor = -2$, and y'' to $2 * y' = 4$. So $(x'', y'', z'') = (-2, 4, -2)$ are the values of (x, y, z) after one iteration.
So $mn - x''y'' = 6 - (-2)(4) = 14 \neq z'' =$.
Thus $z' = mn - x'y' \wedge (x', y', z')$ are the values of (x, y, z) before the loop iteration $\wedge$ (x'', y'', z'') are the values of (x, y, z) after the loop iteration, and $z'' \neq mn - x''y''$.

Since $x, y, z, x', y', z', m, n$ are integers, $\exists x', y', z', x'', y'', z'', m, n \in \mathbb{N}$, $z' = mn - x'y' \wedge (x'', y'', z'')$ are the values of (x, y, z) after one iteration of the loop $\wedge$ $z'' \neq mn - x''y''$.

Complexity of Algorithms

1. Consider the MatrixMultiplication algorithm that multiplies two matrices. Let A and B be 2D arrays, which store the two matrices. The result of the matrix multiplication is stored in a 2D array C. In the code below, $rows$ and $cols$ are the lengths of the rows and columns of the array.

```
MatrixMultiplication(A, B)
1.   if (A.cols != B.rows)
2.      return null
3.   i = 0
4.   while (i < A.rows){
5.      k = 0
6.      while (k < B.cols){
7.         C[i][k] = 0
8.         j = 0
9.         while (j < A.cols){
10.           temp = C[i][k]
11.           result = A[i][j] · B[j][k]
12.           C[i][k] = temp + result
13.           j = j + 1
           }
14.        k = k + 1
        }
15.     i = i + 1
     }
16.  return C
```

(a) Compute the exact number of steps executed for each line of the algorithm.

Soln:

```
Matrix_Multiplication(A, B)
1.   if (A.cols != B.rows) {6 steps}
2.      return NULL      {2 steps}
3.   i := 1              {3 steps}
4.   while (i ≤ A.rows) {5 steps}
5.      k := 1           {3 steps}
6.      while (k ≤ B.cols) {5 steps}
7.         C[i][k] := 0  {5 steps}
8.         j := 1        {3 steps}
9.         while (j ≤ A.cols) {5 steps}
10.           α := C[i][k] {5 steps}
11.           β := A[i][j] · B[j][k] {9 steps}
12.           C[i][k] := α + β {7 steps}
13.           j := j + 1 {5 steps}
14.        k := k + 1    {5 steps}
15.     i := i + 1       {5 steps}
16.  return C            {2 steps}
```

(b) Compute $t_{MM}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 4 & 2 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 1 \\ 2 & 4 & 6 & 8 & 9 \end{bmatrix}\right)$.

Soln:
The test in line 1. takes 6 steps.
The initialization in line 3 takes 3 steps.

The inner while loop from lines *9 – 13* is executed $A.\texttt{cols} = 3$ times. The total number of steps for the execution once of the while loop is $5 + 5 + 9 + 7 + 5 = 31$. Hence the total number of steps spent in the while loop is $3 \cdot 31 + 5 = 98$ (the last 5 steps come from the fact that the while loop condition is checked one final time and fails, which terminates the execution of the loop).

The while loop from lines *6 – 14* is executed $B.\texttt{cols} = 5$ times. The total number of steps for one iteration of this while loop is $5 + 5 + 3+$ (lines *9 – 13*) $+5 = 13 + 98 + 5 = 116$ steps. Hence the total number of steps spent in the while loop is $5 \cdot 116 + 5 = 585$.

The while loop from lines *4 – 15* is executed $A.\texttt{rows} = 4$ times. The total number of steps for the execution once of this while loop is $5 + 3+$ (lines *6 – 14*) $+5 = 8 + 585 + 5 = 598$ steps. Hence the total number of steps spent in the while loop is $4 \cdot 598 + 5 = 2397$.

The total number of steps on the whole function is $6 + 3 + 2397 + 2 = 2408$. So $t_{MM}(A, B) = 2408$.

(c) For $n \times n$ arrays, A and B, find $T_{MM}(n)$. **Soln:**
Assume A and B are both $n \times n$ matrices. We compute the number of steps of the algorithm in this case.

The while loop from lines *9 – 13* will take $n \times 31 + 5 = 31n + 5$ steps.

The while loop from lines *6 – 14* will take $(13 + (31n + 5) + 5)n + 5 = 31n^2 + 23n + 5$ steps.

The while loop from lines *4 – 15* will take $(8+(31n^2+23n+5)+5)n+5 = 31n^3+23n^2+18n+5$ steps.

The whole algorithm takes $6 + 3 + (31n^3 + 23n^2 + 18n + 5) + 2 = 31n^3 + 23n^2 + 18n + 16$.
Hence $T_{MM}(n) = 31n^3 + 23n^2 + 18n + 16$.

(d) For $n \times n$ arrays, A and B, prove $T_{MM}(n) \in \mathcal{O}(n^3)$.

Soln:

Prove that $T_{MM}(n) \in \mathcal{O}(n^3)$:

Let $c = 88$
Then $c \in \mathbb{R}^+$
  Let $B = 1$
  Then $B \in \mathbb{N}$
    Suppose $n \geq B$,
      Then $n \geq 1$
        $16n \geq 16$, so $16n^3 \geq 16$
        $18n \geq 18$, so $18n^2 \geq 18n$, so $18n^3 \geq 18n$
        $23n \geq 23$, so $23n^3 \geq 23n^2$
        $31n^3 \geq 31n^3$
      Therefore $88n^3 \geq 16 + 18n + 23n^2 + 31n^3$
    Hence, $n \geq B \rightarrow 16 + 18n + 23n^2 + 31n^3 \leq c \cdot n^3$

Since n is an arbitrary natural number, $\forall n \in \mathbb{N}, n \geq B \rightarrow 16 + 18n + 23n^2 + 31n^3 \leq c \cdot n^3$
Since c is a positive real number, $\forall c \in \mathbb{R}^+, \exists B \in \mathbb{N}, n \geq B \rightarrow 16 + 18n + 23n^2 + 31n^3 \leq c \cdot n^3$
Since $T_{MM}(n) = 16 + 18n + 23n^2 + 31n^3$, $T_{MM}(n) \in \mathcal{O}(n^3)$

(e) For $n \times n$ arrays, A and B, prove $T_{MM}(n) \in \Omega(n^3)$.
Soln:
From the part(b), we have $T_{MM}(A, B) = 16 + 18n + 23n^2 + 31n^3$.
Prove that $T_{MM}(n) \in \Omega(n^3)$:

Let $c = 31$
Then $c \in \mathbb{R}^+$
Let $B = 1$
Then $B \in \mathbb{N}$
Suppose $n \geq B$,
Then $n \geq 1$
$16 \geq 0$
$18n \geq 0$
$23n^2 \geq 0$
$31n^3 \geq 31n^3$
Therefore $16 + 18n + 23n^2 + 31n^3 \geq 31n^3$
Hence, $n \geq B \rightarrow 16 + 18n + 23n^2 + 31n^3 \geq c \cdot n^3$
Since n is an arbitrary natural number, $\forall n \in \mathbb{N}, n \geq B \rightarrow 16 + 18n + 23n^2 + 31n^3 \geq c \cdot n^3$
Since c is a positive real number, $\forall c \in \mathbb{R}^+, \exists B \in \mathbb{N}, n \geq B \rightarrow 16 + 18n + 23n^2 + 31n^3 \geq c \cdot n^3$
Since $T_{MM}(n) = 16 + 18n + 23n^2 + 31n^3$, $T_{MM}(n) \in \Omega(n^3)$

(f) For $n \times n$ arrays, A and B, is there a natural number $r \in \mathbb{N}$ such that $T_{MM} \in \Theta(n^r)$? Justify your answer.
Soln:
From the two proofs above, $T_{MM}(n) \in \mathcal{O}(n^3)$ and $T_{MM}(n) \in \Omega(n^3)$, so it follows that $T_{MM}(n) \in \Theta(n^3)$. Hence $r = 3$.

2. Consider the Selection Sort algorithm below, that sorts an input array A into non-decreasing order.

```
SS(A):
    i := 0                          // 3 steps
    while i < A.length:             // 5 steps
        s := i                      // 3 steps
        j := i + 1                  // 5 steps
        while j < A.length:         // 5 steps
            if A[j] < A[s]:         // 6 steps
                s := j              // 3 steps
            j := j + 1              // 5 steps
        t := A[i]                   // 4 steps
        A[i] := A[s]                // 5 steps
        A[s] := t                   // 4 steps
        i := i + 1                  // 5 steps
```

For each question below, find the *exact* answer without using $\mathcal{O}$, Ω, or Θ.

(a) Compute the number of steps executed for each line of the algorithm.
Solution: (See above.)

(b) Compute $t_{SS}([4, 2, 1, 3, 5])$.

Solution: Let's analyze the general behaviour of the algorithm, ignoring line 7 for the moment. Let $n = A.\text{length}$.

In general, lines 5,6,8 are executed $n - 1 - i$ times (once for each value of $j = i + 1, i + 2, \ldots, n - 1$), for a total of $16n - 16 - 16i = 16n - 16(i + 1)$ steps. Thus, lines $2, 3, 4, 5, 6, 8, 9, 10, 11, 12$ take $16n - 16(i+1) + 36$ steps (remember that line 5 will get executed one last time for $j = n$, which was not included in the $16n - 16(i + 1)$), and these lines are executed for each value of $i = 0, 1, \ldots, n - 1$ for a total of

$$\begin{aligned}
\sum_{i=0}^{n-1} 16n - 16(i+1) + 36 &= 16n^2 + 36n - 16\sum_{j=1}^{n} j \\
&= 16n^2 + 36n - 16\frac{n(n+1)}{2} \\
&= 16n^2 + 36n - 8n^2 - 8n \\
&= 8n^2 + 28n \text{ steps.}
\end{aligned}$$

When we add the 8 steps for line 1 and the last execution of line 2 (when $i = n$), we get a total of $8n^2 + 28n + 8$ steps.

For the specific input $[4, 2, 1, 3, 5]$, $n = A.\text{length} = 5$ and line 7 actually gets executed a total of 3 times (once for $i = 0, j = 1$, once for $i = 0, j = 2$, and once for $i = 2, j = 3$), which gives a grand total of $8(5^2) + 28(5) + 8 + 9 = 357$ steps.

(c) Compute $T_{SS}(3)$.

Solution: We need to figure out the worst number of times that line 7 gets executed, and add this to the general analysis result of $8n^2+28n+8$ above. Line 7 gets executed at most 3 times (once for $i = 0, j = 1$, once for $i = 0, j = 2$, and once for $i = 1, j = 2$). However, if line 7 gets executed for $i = 0, j = 1$ and $i = 0, j = 2$, it means the original elements are $[a, b, c]$ in order $a > b > c$ so after the first swap in lines 9–11, the array will become $[c, b, a]$ with $c < b < a$ and line 7 will not be executed again. Other cases behave similarly: in practice, line 7 is executed at most twice, for a total of 6 steps.

In addition to the $8n^2+28n+8$ steps from our general analysis, this means that $T_{SS}(3) = 8(3^2)+28(3)+8+6 = 170$.

(d) Compute $T_{SS}(n)$.

Solution: From the general analysis above, we know that $T_{SS}(n) = 8n^2 + 28n + 8+$ the worst-case number of times that line 7 is executed. Although in theory, line 7 could be executed every time through the loop, we know that in practice this cannot happen because if line 7 is executed for every value of $j = i + 1, \ldots, n - 1$ it means that $A[i] > A[i + 1] > \cdots > A[n - 1]$ and $A[i]$ will be swapped with $A[n - 1]$ on lines 9–11 so that line 7 will never be executed again for $j = n - 1$.

The worst-case seems to happen when A is initially sorted in reverse order, in which case line 7 will get executed exactly $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$ times, so that $T_S S(n) = 8n^2 + \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil + 28n + 8$.

However, this is a more difficult problem that it looks. In fact, it is more difficult than I intended: to prove rigorously that the reverse-sorted A is the worst-case is beyond the scope of this assignment and of this course.

3. Prove that $T_{\text{BFT}}(n) \in \Theta(n^2)$, where BFT is the algorithm below.

```
BFT(E, n):
        i := n − 1
        while i > 0:
                P[i] := −1
                Q[i] := −1
```

```
5.          i := i − 1
6.      P[0] := n
7.      Q[0] := 0
8.      t := 0
9.      h := 0
10.     while h ≤ t:
11.         i := 0
12.         while i < n:
13.             if E[Q[h]][i] ≠ 0 and P[i] < 0:
14.                 P[i] := Q[h]
15.                 t := t + 1
16.                 Q[t] := i
17.             i := i + 1
18.         h := h + 1
```

(Although this is not directly relevant to the question, this algorithm carries our a breadth-first traversal of the graph on n vertices whose adjacency matrix is stored in E.)

Solution:

Proof that $T(n) \in \mathcal{O}(n^2)$**:**

Let $c = 80$ and $B = 1$. Then, $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and suppose $n \geq 1$.

From the algorithm, for any input E, n,

line 1 takes 5 steps;

lines 2–5 take 17 steps and are executed exactly $n - 1$ times (once for each value of $i = n-1, n-2, \ldots, 1$), in addition to 4 steps for one last execution of line 2 (when the loop test becomes false) for a total of $17(n-1) + 4 = 17n - 13$ steps;

lines 6–9 take 14 steps;

lines 12–17 take at most 35 steps (if the condition of the **if** statement is true at every iteration) and these lines are executed exactly n times (once for each value of $i = 0, 1, \ldots, n-1$), in addition to 4 steps for one last execution of line 12 (when the loop test becomes false) for a total of at most $35n + 4$ steps;

hence, lines 10–18 take no more than $35n + 4 + 12 = 35n + 16$ steps;

we argue that lines 10–18 are executed at most n times, as follows: from line 10, $h \leq t$; from line 16, $Q[t]$ always gets assigned a value ≥ 0; hence $Q[h]$ is always ≥ 0, so $P[i]$ always gets assigned a value ≥ 0 on line 14; lines 14–16 are executed at most once for each value of $i = 1, \ldots, n-1$ because they are executed only when $P[i] < 0$ and $P[0] = n \geq 0$ (from line 6) and for every value of i such that $P[i] < 0$, line 14 sets $P[i] \geq 0$; therefore, the value of t will always be at most $n - 1$ (t starts at 0 and by the reasoning we just completed, it is incremented at most $n - 1$ times); hence, lines 10–18 are executed at most once for each value of $h = 0, 1, \ldots, n-1$, in addition to 4 steps for one last execution of line 10 (when the loop test becomes false) for a total of no more than $(35n + 16)n + 4 = 35n^2 + 16n + 4$ steps;

therefore, the algorithm takes no more than $5 + 17n - 13 + 14 + 35n^2 + 16n + 4 = 35n^2 + 33n + 10$ steps.

Since $n \geq 1$, $T(n) \leq 35n^2 + 33n + 10 \leq 35n^2 + 33n^2 + 10n^2 \leq 80n^2$.

Hence, $n \geq 1 \rightarrow T(n) \leq 80n^2$ and since n is an arbitrary natural number, $\forall n \in \mathbb{N}, n \geq 1 \rightarrow T(n) \leq 80n^2$.

Therefore, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow T(n) \leq cn^2$, *i.e.*, $T(n) \in \mathcal{O}(n^2)$.

Proof that $T(n) \in \Omega(n^2)$**:**

Let $c = 1$ and $B = 1$. Then, $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and suppose $n \geq 1$.

Consider an input E, n such that $E[i][j] = 1$ for all indices $0 \leq i < n, 0 \leq j < n$. The first time that lines 12–17 are executed, the condition of the **if** statement will be true for all values of $i = 1, 2, \ldots, n-1$ so at the end of the loop, t will have value $n-1$ (since t starts with value 0 and gets incremented $n-1$ times). This means that lines 10–18 will get executed for every value of $h = 0, 1, \ldots, n-1$. Since lines 12–17 always get executed exactly n times (once for each value of $i = 0, 1, \ldots, n-1$), they take at least n steps. Since lines 10–18 get executed n times, and they take at least n steps for each iteration, they take at least n^2 steps in total. So $T(n) \geq n^2$.

Hence, $n \geq 1 \rightarrow T(n) \geq n^2$ and since n is an arbitrary natural number, $\forall n \in \mathbb{N}, n \geq 1 \rightarrow T(n) \geq n^2$.

Therefore, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow T(n) \geq cn^2$, *i.e.*, $T(n) \in \Omega(n^2)$.

Hence, $T(n) \in \Theta(n^2)$, as required.

IMPORTANT NOTE: Make sure that you understand how to do the scratch work to figure out the values of the constants. If you have any questions about this, ask us!

4. Your rich uncle gives you some money, so you decide to drop CSCA65 and travel around Europe for the summer. There's a list of n cities $c_1, \ldots, c_n$ that you want to visit, but you only have enough money for $n-2$ train rides, so you're going to have to walk for one leg of the trip. Seeing as it's summer and it's hot outside, you want to walk for the shortest distance possible, so you want to find the two cities which are closest together. Assume you have a 2 dimensional $n \times n$ array $dist$ such that $dist(c_i, c_j)$ returns the distance between any two cities.

   Here is an iterative algorithm for finding the closest pair:

```
Find_closest_pair (c_1, c_2, ..., c_n)              Steps:

    if (n = 1) return NIL                             6
    min := infinity                                   3
    min_i := 0                                        3
    min_j := 0                                        3
    i := 1                                            3
    while (i <= n)                                    4
        j := i+1                                      5
        while (j <= n)                                4
           if (dist(c_i, c_j) < min) then             7
               min := dist(c_i, c_j)                  5
               min_i := i                             3
               min_j := j                             3
           j := j+1                                   5
        i := i+1                                      5
    return (min_i, min_j)                             3

end
```

   (a) Fill in the number of steps for each line above, you may assume that accessing `dist` takes one step as in 1 dimensional arrays.

   (b) Let $T_{CP}(n)$ represent the worst case complexity of `Find\_Closest\_Pair(c_1,c_2,c_3,c_4)`. Compute $T_{CP}(4)$ exactly.
   b239

   (c) Determine a function g such that $T_{CP}(c_1, c_2, \ldots, c_n) \in \Theta(g(n))$ and prove that your g is correct.
   $g(n) = n^2$. We need to prove the $T_{CP}(c_1, c_2, \ldots, c_n) \in O(g(n))$ and $T_{CP}(c_1, c_2, \ldots, c_n) \in \Omega(g(n))$.
   First let us prove O:

The lines 1..5 and the line 15 are executed only once, they require in total 21 steps.
The lines of the outer loop: 6 7 and 14 are executed at most n times, then they require at most 27n steps.
The lines in the inner loop, are executed at most n times at each iteration of the outer loop, so require at most 14n steps at each iteration of the outer loop. The outer loop is executed at most n times, the lines of the inner loop are then executed at most $14n^2$.
Therefore this algorithm requires less than $14n^2 + 27n + 21$, thus $T_{CP}(n) \leq 4n^2 + 27n + 21$.
Let us now do the formal proof:

Let c = 62
  Let n_0 = 1
    Let $n \in N$
      Assume $n > n_0$
        then $n > 1$
        thus $n^2 > n$ so $27n^2 \geq 27n$
        and also $n^2 > 1$ so $21n^2 \geq 21$
        then $14n^2 + 27n^2 + 21n^2 \geq 14n^2 + 27n + 21 \geq T_{CP}(n)$
        thus $T_{CP}(n) \leq 62n^2$ so $T_{CP}(n) \leq cn^2$
      therefore $n > n_0 \Rightarrow T_{CP}(n) \leq cn^2$
    Since n is an arbitrary number in N, $\forall n \in N n > n_0 \Rightarrow T_{CP}(n) \leq cn^2$
Since c and n_0 are in N, $\exists c \in N \exists n_0 \in N \forall n \in N n > n_0 \Rightarrow T_{CP}(n) \leq cn^2$

Therefore $T_{CP}(n) \in O(n^2)$

We now need to prove that $T_{CP}(n) \in \Omega(n^2)$.
At iteration k of the outer loop the inner loop will be executed $n-i$ times. The inner loop contain more than 2 steps, so at iteration k of the outer loop, the inner loop would require more than $2(n-k)$ steps. So the iteration k of the outer loop will require more than $2(n-k)+1$ steps. The outer loop is executed n times, so the total number of steps for the outer loop will be bigger than $\Sigma_{k=1}^{n}(2(n-k)+1)$. The other lines are executed only once, and then require a constant number of steps. Therefore $T_{CP}(n) \geq \Sigma_{k=1}^{n}(2(n-k)+1)+1$.
Let us now do the formal proof:

Let c = 1
  Let n_0 = 1
    Let $n \in N$
      Assume $n > n_0$
        then $n > 1$
        $T_{CP}(n) \geq \Sigma_{k=1}^{n}(2(n-k)+1)+1$
        so $T_{CP}(n) \geq \Sigma_{l=0}^{n-1}(2l+1)+1$
        so $T_{CP}(n) \geq 2\Sigma_{l=0}^{n-1} l + n + 1$
        so $T_{CP}(n) \geq 2\frac{(n-1)n}{2} + n + 1$
        so $T_{CP}(n) \geq n^2 + 1$
        then $T_{CP}(n) \geq n^2$
        thus $T_{CP}(n) \geq cn^2$
      therefore $n > n_0 \Rightarrow T_{CP}(n) \geq cn^2$
    Since n is an arbitrary number in N, $\forall n \in N n > n_0 \Rightarrow T_{CP}(n) \geq cn^2$
Since c and n_0 are in N, $\exists c \in N \exists n_0 \in N \forall n \in N n > n_0 \Rightarrow T_{CP}(n) \geq cn^2$

Therefore $T_{CP}(n) \in \Omega(n^2)$
Thus $T_{CP}(n) \in \Theta(n^2)$

Asymptotic Notation

1. (a) First we prove that $\log_2 x = \log_{10} x \log_2 10$:
   Let $a = \log_2 x$, $b = \log_{10} x$
   Then $2^a = x = 10^b$ (by defn of log)
   Thus $a \log_2 2 = b \log_2 10$ (take log of both sides)
   Thus $\log_2 x = \log_{10} x \log_2 10$ (substitute in for a and b)

   Now we need to prove the statement:
   $\exists c \in \mathbb{R}^+ \log_2 x = c \log_{10} x$
   Let $c = \log_2 10$. Then $c \in \mathbb{R}^+$
   Then $\log_2 x = c \log_{10} x$ (by proof above)
   Therefore $\exists c \in \mathbb{R}^+ \log_2 x = c \log_{10} x$.
   (6 marks - 4 for proof, 2 for structure)

   (b) We can infer that $f(x) \in \Theta(g(x)$ and $g(x) \in \Theta(f(x))$.
   (5 marks)

   (c) $123_{(10)} = 1111011_{(2)}$, $177_{(10)} = 10110001_{(2)}$
   $1101011_{(2)} = 107_{(10)}$, $1000011_{(2)} = 67_{(10)}$
   (4 marks)

   (d) $1100 \times 101 = 111100$. To compute $9x$ where x is binary, and working only in binary, one would convert $9_{(10)} = 1001_{(2)}$. Then using binary long-hand multiplication, we see that multiplying any number by the binary representation of 9 is equivalent to adding the number x to x right-shifted by 3 digits. $9_{(10)} \times 110110_{(2)} = 111100110_{(2)}$. (5 marks)

2. (a) We will prove the statement to be true.
   Thus we will prove $\exists d \in \mathbb{R}^+, \exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow 6n^5 + n^2 - n^3 \leq dn^5$
   Let $d = 6$, then $c \in \mathbb{R}^+$
   Let $a = 1$, then $a \in \mathbb{N}$
   Suppose $n \geq a$
   Then $6n^3 + 1 \leq 6n^3$
   Then $(6n^3 + 1)n^2 \leq 6n^5$
   Then $6n^5 + n^2 \leq 6n^5$
   Then $6n^5 + n^2 - n^3 \leq 6n^5$
   Then $6n^5 + n^2 - n^3 \leq cn^5$
   Thus $n \geq a \rightarrow 6n^5 + n^2 - n^3 \leq dn^5$.
   Thus $\forall n \in \mathbb{N}, n \geq a \rightarrow 6n^5 + n^2 - n^3 \leq dn^5$
   . Thus $\exists d \in \mathbb{R}^+, \exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow 6n^5 + n^2 - n^3 \leq dn^5$
   . (6 marks)

   (b) We will prove the statement to be true.
   Thus we will prove $\exists d \in \mathbb{R}^+, \exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow (1 + 1/n) \geq d/n$
   Let $d = 1$, then $c \in \mathbb{R}^+$
   Let $a = 1$, then $a \in \mathbb{N}$
   Suppose $n \geq a$
   Then $1 + 1/n \geq 1/n$
   Thus $n \geq a \rightarrow (1 + 1/n) \geq d/n$
   Thus $\forall n \in \mathbb{N}, n \geq a \rightarrow (1 + 1/n) \geq d/n$
   Thus $\exists d \in \mathbb{R}^+, \exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow (1 + 1/n) \geq d/n$
   (7 marks)

   (c) We will prove the statement to be true.
   Thus we will prove $\exists d \in \mathbb{R}^+, \exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow (n \log n) \leq dn^2$

Let $d = 1$, then $c \in \mathbb{R}^+$
Let $a = 2$, then $a \in \mathbb{N}$
Suppose $n \geq a$
Then $\log n \leq n$
Then $n \log n \leq n^2$
Thus $n \geq a \rightarrow (n \log n) \leq dn^2$
Thus $\forall n \in \mathbb{N}, n \geq a \rightarrow (n \log n) \leq dn^2$
Thus $\exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow (n \log n) \leq dn^2$
Thus $\exists d \in \mathbb{R}^+, \exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow (n \log n) \leq dn^2$
(7 marks)

(d) We will prove that $(\log_2 n)^n \in \mathcal{O}(n^{n \log_2 n})$
Thus we will prove $\exists d \in \mathbb{R}^+, \exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow (\log_2 n)^n \leq dn^{n \log_2 n}$
We can ignore the base on the logs.
Let $d = 1$, then $c \in \mathbb{R}^+$
Let $a = 2$, then $a \in \mathbb{N}$
Suppose $n \geq a$
Then $\log(\log(n) \leq \log n$ (since $\log n \leq n$ for $n \geq 1$).
Then $\log(\log(n)) \leq (\log n)(\log n)$
Then $n \log(\log(n)) \leq n(\log n)(\log n)$
Then $\log((\log n)^n) \leq \log(n^{n \log n})$ (by rule of log/exponents)
Then $(\log n)^n \leq n^{n \log n}$ (exponentiate)
Then $(\log_2 n)^n \leq n^{n \log_2 n}$
Thus $n \geq a \rightarrow (\log_2 n)^n \leq dn^{n \log_2 n}$
Thus $\forall n \in \mathbb{N}, n \geq a \rightarrow (\log_2 n)^n \leq dn^{n \log_2 n}$
Thus $\exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow (\log_2 n)^n \leq dn^{n \log_2 n}$
Thus $\exists d \in \mathbb{R}^+, \exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow (\log_2 n)^n \leq dn^{n \log_2 n}$
(bonus marks)

3. The error in the argument occurs in ignoring the fact that k is a function of n. Therefore, what we are really getting is $\sum_{k=1}^{n} kn = n \sum_{k=1}^{n} k = n \cdot \frac{n(n-1)}{2} \in \mathcal{O}(n^3)$.
(5 marks)

Thus we will prove that $\sum_{k=1}^{n} kn \in \mathcal{O}(n^3)$, that is,
$\exists d \in \mathbb{R}^+, \exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow \sum_{k=1}^{n} kn \leq dn^3$
Let $d = 1$, then $c \in \mathbb{R}^+$
Let $a = 1$, then $a \in \mathbb{N}$
Suppose $n \geq a$
Then $n^3/2 \leq n^3$
Thus $n^3/2 - n^2/2 \leq n^3$
And $n \cdot \frac{n(n-1)}{2} = n^3/2 - n^2/2$
And $\sum_{k=1}^{n} kn = n \cdot \frac{n(n-1)}{2}$
Thus $\sum_{k=1}^{n} kn \leq n^3$
Thus $n \geq a \rightarrow \sum_{k=1}^{n} kn \leq dn^3$
Thus $\forall n \in \mathbb{N}, n \geq a \rightarrow \sum_{k=1}^{n} kn \leq dn^3$
Thus $\exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow \sum_{k=1}^{n} kn \leq dn^3$
Thus $\exists d \in \mathbb{R}^+, \exists a \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq a \rightarrow \sum_{k=1}^{n} kn \leq dn^3$
(15 marks)

4. (a) Proof:

Suppose $\lim_{n\to\infty} f(n)/g(n) = p$
(if the limit does not exist, then the proof is trivial).

Then for some value $n_0 \in \mathbb{N}$, if $n \geq n_0$, then there is a c such that $|f(n)/g(n) - p| \leq c$. Suppose that $n \geq n_0$.

Then $p - c \leq \frac{f(n)}{g(n)} \leq p + c$
Also notice that if the limit exists, it must be true that
for some $a \in \mathbb{N}$, that if $n \geq a$,
then $f(n)$ and $g(n)$ must either both be positive, or both be negative.
(otherwise the function $f(n)/g(n)$ would oscillate between
negative and positive and the limit could not exist).
Thus $(p - c)g(n) \leq f(n) \leq (p + c)g(n)$
(This works if g(n) is positive - if it is negative,
then flip the inequality signs around and continue).
Thus $n \geq n_0 \rightarrow (p - c)g(n) \leq f(n) \leq (p + c)g(n)$
Now we will prove that:
$\exists d_1, d_2 \in \mathbb{R}^+, \exists b \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq b \rightarrow (f(n) \leq d_1 g(n) \wedge f(n) \geq d_2 g(n))$
. Let $b = \max(a, n_0)$, then $b \in \mathbb{N}$.
Let $d_1 = p + c$, $d_2 = p - c$ then $d_1, d_2 \in \mathbb{R}^+$.
Suppose $n \geq b$
Then $d_2 g(n) \leq f(n) \leq d_1 g(n)$
Then $f(n) \leq d_1 g(n) \wedge f(n) \geq d_2 g(n)$
Then $n \geq b \rightarrow (f(n) \leq d_1 g(n) \wedge f(n) \geq d_2 g(n))$
Thus $\exists d_1, d_2 \in \mathbb{R}^+, \exists b \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq b \rightarrow (f(n) \leq d_1 g(n) \wedge f(n) \geq d_2 g(n))$
Thus $f(n) \in \Theta(g(n))$
Thus $\lim_{n\to\infty} f(n)/g(n) = p \rightarrow f(n) \in \Theta(g(n))$.
Thus $f(n) \in \Theta(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = p$
(15 marks)

(b) What does it mean if $p = 0$? If $p = 0$ then $g(n)$ grows asymptotically faster than $f(n)$, so we know that $f(n) \in \mathcal{O}(g(n))$.
(5 marks)

5. From the definitions of $\mathcal{O}$, Ω and Θ, prove or disprove each of the following using our carefully structured form:

(a) $n! \in \Omega(n^n)$
Soln:
This is false, so we want to show $n! \notin \Omega(n^n)$.
Negate the definition of Ω to get: $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge g(n) \leq cf(n)$.
Let $c \in \mathbb{R}^+$
Let $B \in \mathbb{N}$
Let $n = max\{B + \lceil c \rceil, B + \lceil \frac{1}{c} \rceil\}$
Then $n \in \mathbb{N}$
So $n \geq B$ (since $n = B + \lceil c \rceil and \lceil c \rceil \geq 1$)
$n \leq n$ (By algebra)
$n(n-1) \leq n^2$ (Since $n \geq 0$)
$n(n-1)(n-2) \leq n^3$
...
$n(n-1)(n-2)...(n-(n-2))(1) \leq n^n$
Therefore, $n! \leq n^n$
Case 1: $c \geq 1$
$n! \leq cn^n$ (Since $c \geq 1$)
Case 2: $0 < c < 1$
$n \geq \frac{1}{c}$ (by defn of n), so $nc \geq 1$

Therefore, $cn^n \geq n^{n-1}$.
$n \leq n$ (By algebra)
$n(n-1) \leq n^2$ (Since $n \geq 0$)
$n(n-1)(n-2) \leq n^3$
...
$n(n-1)(n-2)...(n-(n-2)) \leq n^{n-1}$
Also, $n(n-1)(n-2)...(n-(n-2)) = n(n-1)(n-2)...(n-(n-2))(1) = n!$
Hence, $n! \leq cn^n$ (Since $cn^n \geq n^{n-1}$)
Since n is a natural number, $\exists n \in \mathbb{N}, n \geq B \wedge n! \leq cn^n$
Since B is an arbitrary natural number, $\forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n! \leq cn^n$
Since c is an arbitrary positive real number, $\exists c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n! \leq cn^n$

(b) $\frac{n+12}{2n^2+2} \in \mathcal{O}(1)$
Soln:
This is true. By definition of $\mathcal{O}$, show: $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow \frac{n+12}{2n^2+2} \leq c$
Let $c = 1$
Then $c \in \mathbb{R}^+$
Let $B = 3$
Then $B \in \mathbb{N}$
Let $n \in \mathbb{N}$
Suppose $n \geq B$
$n \geq 3$ (since $B = 3$)
$n^2 \geq 3n$
$2n^2 \geq 6n$
$2n^2 + 2 \geq 6n + 2$
$1 \geq \frac{6n+2}{2n^2+2}$ (Since $n \geq 0$)
$6n + 2 \geq n + 12$ (since $n \geq 3$)
Therefore, $1 \geq \frac{n+12}{2n^2+2}$
Hence, $n \geq B \rightarrow 1 \geq \frac{n+12}{2n^2+2}$
Since n is an arbitrary natural number, $\forall n \in \mathbb{N}, n \geq B \rightarrow 1 \geq \frac{n+12}{2n^2+2}$
Since B is a natural number, $\exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow 1 \geq \frac{n+12}{2n^2+2}$
Since c is a positive real number, $\forall c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow c \geq \frac{n+12}{2n^2+2}$

(c) $n^3 + 3n \in \Theta(2n^3)$
Soln:
This is true. By the definition of Θ, show:
$\exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow c_1(2n^3) \leq (n^3 + 3n) \leq c_2(2n^3)$

Let $c_1 = \frac{1}{2}$
Then $c_1 \in \mathbb{R}^+$
Let $c_2 = 1$
Then $c_2 \in \mathbb{R}^+$
Let $B = 2$
Then $B \in \mathbb{N}$
Let $n \in \mathbb{N}$
Suppose $n \geq B$
Then $n \geq 2$
$n^3 \leq n^3 + 3n$ (By $n \geq 2$)
Thus $c_1 2n^3 \leq (n^3 + 3n)$, (since $c_1 = \frac{1}{2}$)
$3n \leq n^3$, since $n \geq 2$

Thus $n^3 + 3n \le n^3 + n^3$, so $(n^3 + 3n) \le c_2 2n^3$ (By $c_2 = 1$)
Hence, $c_1(2n^3) \le (n^3 + 3n) \le c_2(2n^3)$
Therefore $n \ge B \to c_1(2n^3) \le (n^3 + 3n) \le c_2(2n^3)$
Since n is an arbitrary natural number, $\forall n \in \mathbb{N}, n \ge B \to c_1(2n^3) \le (n^3 + 3n) \le c_2(2n^3)$
Since B is a natural number, $\exists B \in \mathbb{N}, n \ge B \to c_1(2n^3) \le (n^3 + 3n) \le c_2(2n^3)$
Since c_2 is a positive real number, $\exists c_1 \in \mathbb{R}^+, \exists B \in \mathbb{N}, n \ge B \to c_1(2n^3) \le (n^3 + 3n) \le c_2(2n^3)$
Since c_1 is a positive real number, $\exists c_1 \in \mathbb{R}^+, \exists c_1 \in \mathbb{R}^+, \exists B \in \mathbb{N}, n \ge B \to c_1(2n^3) \le (n^3 + 3n) \le c_2(2n^3)$

6. For any function $f : \mathbb{N} \to \mathbb{R}^{\ge 0}$, define the function $\bar{f} : \mathbb{N} \to \mathbb{R}^{\ge 0}$ as follows:

$$\forall n \in \mathbb{N}, \bar{f}(n) = \lceil f(n) \rceil$$

For any function $f : \mathbb{N} \to \mathbb{R}^{\ge 0}$, define the function $\hat{f} : \mathbb{N} \to \mathbb{R}^{\ge 0}$ as follows:

$$\forall n \in \mathbb{N}, \hat{f}(n) = \lfloor f(n) \rfloor$$

Prove or disprove each of the following using our carefully structured form:

(a) $f \in \mathcal{O}(g) \leftrightarrow f \in \mathcal{O}(\bar{g})$ **Soln:**
False. The implication $f \in \mathcal{O}(\bar{g}) \to f \in \mathcal{O}(g)$ is not true. We will give a counterexample.
Let $\forall n \in \mathbb{N}, f(n) = 2$, and $\forall n \in \mathbb{N}, g(n) = \frac{1}{n+1}$. We show that $f \in \mathcal{O}(\bar{g})$ but $f \notin \mathcal{O}(g)$.

Let $c = 2$
Then $c \in \mathbb{R}^+$
Let $B = 0$
Then $B \in \mathbb{N}$
Let $n \in \mathbb{N}$
Suppose $n \ge 0$
$f(n) = 2$ (by the defn of $f(n)$)
$\bar{g}(n) = 1$ (by the defn of $\bar{g}(n)$)
Hence $n \ge 0 \to f(n) \le 2 \cdot \bar{g}(n)$
Hence $\forall n \in \mathbb{N}, n \ge 0 \to f(n) \le 2 \cdot \bar{g}(n)$
Since B is a natural number, $\exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \ge B \to f(n) \le c \cdot \bar{g}(n)$
Since c is a non-negative real number, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \ge B \to f(n) \le c \cdot \bar{g}(n)$

Hence $f \in \mathcal{O}(\bar{g}(n))$.

To show that $f \notin \mathcal{O}(g)$ we negate the definition of $f \in \mathcal{O}(g)$ and prove that the negation is true. So assume that
$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \ge B \land f(n) > c \cdot g(n)$.

Let $c \in \mathbb{R}^+$
Let $B \in \mathbb{N}$
Let $n = \lceil \max\{B, c\} \rceil$
Then $n \in \mathbb{N}$
So $n \ge B$

$f(n) = 2$ (by defn of $f(n)$)
$c \cdot g(n) = c \cdot \frac{1}{n+1} \leq 1$ (since $n \geq c$)
Hence $2 > c \cdot \frac{1}{n+1}$
So $f(n) > c \cdot g(n)$
Therefore, $n \geq B \wedge f(n) > c \cdot g(n)$
Since n is a natural number, $\exists n \in \mathbb{N}, n \geq B \wedge f(n) > c \cdot g(n)$
Since B is an arbitrary natural number, $\forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge f(n) > c \cdot g(n)$
Since c is an arbitrary positive real number, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge f(n) > c \cdot g(n)$

Hence $f \notin \mathcal{O}(g(n))$.

Therefore, $\neg(f \in \mathcal{O}(\bar{g}) \rightarrow f \in \mathcal{O}(g))$, so $\neg(f \in \mathcal{O}(g) \leftrightarrow f \in \mathcal{O}(\bar{g}))$.

(b) $\bar{f} \cdot \hat{f} \in \mathcal{O}(f^2)$

(c) Consider $\mathcal{O}_2(g) = \{f : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0} | \exists c \in R^+, \forall n \in \mathbb{N}, f(n) \leq cg(n)\}$.

Prove or disprove: $f \in \mathcal{O}(g) \leftrightarrow \bar{f} \in \mathcal{O}_2(\bar{g})$. **Soln:**

False. We show a case where $\bar{f} \in \mathcal{O}_2(\bar{g})$ is true, but $f \in \mathcal{O}(g)$ is false.
Let $f(n) = \frac{1}{n}$ and $g(n) = \frac{1}{n^2}$.

Let $c = 1$
Then $c \in \mathbb{R}^+$
Let $n \in \mathbb{N}$
$\bar{f}(n) = 1$ (by defn of $\bar{f}(n)$)
$\bar{g}(n) = 1$ (by defn of $\bar{g}(n)$)
Hence $\bar{f}(n) \leq 1 \cdot \bar{g}(n)$
Since n is an arbitrary natural number, $\forall n \in \mathbb{N}, \bar{f}(n) \leq c \cdot \bar{g}(n)$
Since c is a positive real number, $\exists c \in \mathbb{R}^+, \forall n \in \mathbb{N}, \bar{f}(n) \leq c \cdot \bar{g}(n)$

Hence $\bar{f} \in \mathcal{O}_2(\bar{g})$

Show that $f \notin \mathcal{O}(g)$.

Let $c \in \mathbb{R}^+$
Let $B \in \mathbb{N}$
Let $n = \max\{B, c\} + 1$.
Then $n \in \mathbb{N}$.
Thus $n \geq B$.
$f(n) = \frac{1}{n}$ and $g(n) = \frac{1}{n^2}$ (by defn of $f(n)$ and $g(n)$)
$n \geq c$, so $\frac{1}{n} > c\frac{1}{n^2}$
Hence $f(n) > cg(n)$
Therefore, $n \geq B \wedge f(n) > cg(n)$.
Since n is a natural number, $\exists n \in \mathbb{N}, n \geq B \wedge f(n) > cg(n)$.
Since B is an arbitrary natural number, $\forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge f(n) > cg(n)$.
Since c is a positive real number, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge f(n) > cg(n)$.

Hence $f \notin \mathcal{O}(g)$

7. Prove or disprove each statement below.

(a) $3n^2 - 4n \in \Omega(n^2)$

Soln. We prove that $3n^2 - 4n \in \Omega(n^2)$.

Let c = 1 and B = 2. Then, $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and suppose $n \geq 2$.

$n \geq 2$
$\rightarrow 2n \geq 4$
$\rightarrow 3n \geq n + 4$
$3n - 4 \geq n$
$n(3n - 4) \geq n^2$
$3n^2 - 4n \geq n^2$

So $n \geq 2 \rightarrow 3n^2 - 4n \geq n^2$ and since n is arbitrary,

$\forall n \in \mathbb{N}, n \geq 2 \rightarrow 3n^2 - 4n \geq n^2$.

Hence, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow 3n^2 - 4n \geq cn^2$,

i.e., $3n^2 - 4n \in \Omega(n^2)$.

(b) $n \log n + 5n \in \mathcal{O}(n \log n)$ For this question, assume that "log" means "log_2" (any fixed base was OK).

We prove that $nlogn + 5n \in \mathcal{O}(nlogn)$.

Let c = 6 and B = 2. Then $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and suppose $n \geq 2$.

$n \geq 2 \rightarrow logn \geq log2 = 1$
$\rightarrow nlogn \geq n$
$\rightarrow 5nlogn \geq 5n$
$\rightarrow 6nlogn \geq 5n + nlogn$

So $n \geq 2 \rightarrow nlogn + 5n \leq 6nlogn$

Since n is arbitrary, $\forall n \in \mathbb{N}, n \geq 2 \rightarrow nlogn + 5n \leq 6nlogn$.

Hence,

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow nlogn + 5n \leq cnlogn$$

, i.e., $nlogn + 5n \in \mathcal{O}(nlogn)$.

(c) $200^{10^50} + n/4 \in \mathcal{O}(n)$

We prove that $200^{10^50} + n/4 \in \mathcal{O}(n)$.

Let c = 1.25 and $B = 200^{10^50}$. Then $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and suppose $n \geq 200^{10^50}$.

$n \geq 200^{10^50}$
$\rightarrow n + n/4 \geq 200^{10^50} + n/4$
$\rightarrow 1.25n \geq 200^{10^50} + n/4$

So $n \geq 200^{10^50} \rightarrow 200^{10^50} + n/4 \leq 1.25n$ and since n is arbitrary,

$$\forall n \in \mathbb{N}, n \geq 200^{10^50} \rightarrow 200^{10^50} + n/4 \leq 1.25n.$$

Hence, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow 200^{10^50} + n/4 \leq cn$, i.e.,

$200^{10^50} + n/4 \in \mathcal{O}(n)$.

(d) $5/n + 1 \in \mathcal{O}(1/n)$

(d) We disprove that $5/n + 1 \in \mathcal{O}(1/n)$.

Let $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.

Let $n = max(B, ceil(c))$. Then $n \in \mathbb{N}$.

By definition of max, $n \geq B$.
Also, $n \geq c \rightarrow n > c - 5 \rightarrow 5 + n > c \rightarrow 5/n + 1 > c/n$.
Hence, $\exists n \in \mathbb{N}, n \geq B \wedge 5/n + 1 > c1/n$.
Since c and B are arbitrary, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B / \; 5/n + 1 > c1/n$, i.e.,

$$5/n + 1 not \in \mathcal{O}(1/n).$$

(e) $\forall a \in \mathbb{R}^+, \forall b \in \mathbb{R}^+, \log_a n \in \Theta(\log_b n)$ [Note that there was an error in the original statement: both a and b need to be larger than 1 for the statement to make sense; otherwise, $log_a n$ and $log_b n$ do not take on only positive values.]

8. Prove or disprove the following statement:

For any functions $f : \mathbb{N} \rightarrow \mathbb{R}^+$, $g : \mathbb{N} \rightarrow \mathbb{R}^+$, and $h : \mathbb{N} \rightarrow \mathbb{R}^+$,
if $f \in \mathcal{O}(h)$ and $g \in \mathcal{O}(h)$, then $f \in \mathcal{O}(g)$.

Solution: We disprove the statement.
Let $f(n) = n^2$, $g(n) = n$, and $h(n) = n^3$. Then, $f \in \mathcal{O}(h)$ and $g \in \mathcal{O}(h)$ but $f \notin \mathcal{O}(g)$.

$f \in \mathcal{O}(h)$**:**
Let $c = 1$. Then, $c \in \mathbb{R}^+$. Let $B = 1$. Then, $B \in \mathbb{N}$.
Let $n \in \mathbb{N}$. Suppose that $n \geq 1$.
Because $n \geq 1$, $n^3 \geq n$.
Hence, $n \geq 1 \rightarrow n \leq n^3$ and since n is an arbitrary natural number, $\forall n \in \mathbb{N}, n \geq 1 \rightarrow n \leq n^3$.
Therefore, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow n \leq c \cdot n^3$.

$g \in \mathcal{O}(h)$**:**
Let $c = 1$. Then, $c \in \mathbb{R}^+$. Let $B = 1$. Then, $B \in \mathbb{N}$.
Let $n \in \mathbb{N}$. Suppose that $n \geq 1$.
Because $n \geq 1$, $n^3 \geq n^2$.
Hence, $n \geq 1 \rightarrow n^2 \leq n^3$ and since n is an arbitrary natural number, $\forall n \in \mathbb{N}, n \geq 1 \rightarrow n^2 \leq n^3$.
Therefore, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow n^2 \leq c \cdot n^3$.

$f \notin \mathcal{O}(g)$**:**
Let $c \in \mathbb{R}^+$. Let $B \in \mathbb{N}$.
Let $n = B + \lceil c \rceil + 1$. Then, $n \in \mathbb{N}$.
Because $c \in \mathbb{R}^+$, $\lceil c \rceil \in \mathbb{N}$ so $n \geq B$. Because $\lceil c \rceil + 1 > c$, $n > c$ so $n^2 > c \cdot n$.
Hence, $\exists n \in \mathbb{N}, n \geq B \wedge n^2 > c \cdot n$.
Therefore, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge n^2 > c \cdot n$.

9. Prove or disprove the following statement:

For any functions $f : \mathbb{N} \rightarrow \mathbb{R}^+$, $g : \mathbb{N} \rightarrow \mathbb{R}^+$, and $h : \mathbb{N} \rightarrow \mathbb{R}^+$,
if $f \in \Theta(h)$ and $g \in \Theta(h)$, then $f \in \Theta(g)$.

Soln:

Let f, g, h be functions from $\mathbb{N}$ to $\mathbb{R}^+$.
Suppose that $f \in \Theta(h)$ and $g \in \Theta(h)$.
From the definition, let $c_f \in \mathbb{R}^+, c'_f \in \mathbb{R}^+, B_f \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_f \rightarrow c_f h(n) \leq f(n) \leq c'_f h(n)$.
From the definition, let $c_g \in \mathbb{R}^+, c'_g \in \mathbb{R}^+, B_g \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_g \rightarrow c_g h(n) \leq g(n) \leq c'_g h(n)$.

Let $c_1 = c_f/c'_g, c_2 = c'_f/c_g, B = max(B_f, B_g)$.
Then, $c_1 \in \mathbb{R}^+, c_2 \in \mathbb{R}^+$, and $B \in \mathbb{N}$.
Let $n \in \mathbb{N}$, and suppose that $n \geq B$.
Then,
$g(n) \leq c'_g h(n) [sincen \geq B_g]$
$\rightarrow 1/c'_g g(n) \leq h(n)$
$\rightarrow c_f/c'_g g(n) \leq c_f h(n)$
$\rightarrow c_1 g(n) \leq f(n) [sincen \geq B_f]$
Also, $c_g h(n) \leq g(n) [sincen \geq B_g]$
$\rightarrow h(n) \leq 1/c_g g(n)$
$\rightarrow c'_f h(n) \leq c'_f/c_g g(n)$
$\rightarrow f(n) \leq c_2 g(n)$[since$n \geq B_f$]
So $n \geq B \rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$ and since n is an arbitrary natural number,

$$\forall n \in \mathbb{N}, n \geq B \rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n).$$

Hence, $\exists c_1 inR^+, \exists c_2 inR^+, \exists B \in \mathbb{N}$,
$\forall n \in \mathbb{N}, n \geq B \rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$.
Therefore, $(f \in \Theta(h) \wedge g \in \Theta(h)) \rightarrow f \in \Theta(g)$
and since f, g, h are arbitrary functions from $\mathbb{N}$ to $\mathbb{R}^+$,

for any f, g, h from $\mathbb{N}$ to $\mathbb{R}^+$,

if $f \in \Theta(h)$ and $g \in \Theta(h)$, then $f \in \Theta(g)$.

10. Prove that for any $k \in \mathbb{N}$ and any $a_k \in \mathbb{R}^+, a_{k-1} \in \mathbb{R}^+, \ldots, a_1 \in \mathbb{R}^+, a_0 \in \mathbb{R}^+$,

$$a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0 \in \Theta(n^k).$$

Let $k \in \mathbb{N}$ and $a_k \in \mathbb{R}^+, a_{k-1} \in R^+, ..., a_1 \in R^+, a_0 \in \mathbb{R}^+$.
Let $c_1 = a_k, c_2 = a_k + a_{k-1} + ... + a_1 + a_0, B = 1$
Then, $c_1 \in \mathbb{R}^+, c_2 \in \mathbb{R}^+$, and $B \in \mathbb{N}$.
Let $n \in \mathbb{N}$ and suppose $n \geq 1$.
Since $a_{k-1} > 0, ..., a_1 > 0, a_0 > 0$, we have:
$a_k n^k \leq a_k n^k + a_{k-1} n^{k-1} + ... + a_1 n + a_0$.
Because $n \geq 1$, we have:
$a_k n^k + a_{k-1} n^{k-1} + ... + a_1 n + a_0$
$\leq a_k n^k + a_{k-1} n^k + ... + a_1 n^k + a_0 n^k$
$= (a_k + a_{k-1} + ... + a_1 + a_0) n^k$.
So $n \geq 1 \rightarrow c_1 n^k \leq a_k n^k + ... + a_1 n + a_0 \leq c_2 n^k$
Since n is an arbitrary natural number,

$$\forall n \in \mathbb{N}, n \geq 1 \rightarrow c_1 n^k \leq a_k n^k + ... + a_0 \leq c_2 n^k$$

Hence, $\exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \rightarrow c_1 n^k \leq a_k n^k + ... + a_1 n + a_0 \leq c_2 n^k$.
Since $k, a_k, a_{k-1}, ..., a_1, a_0$ are arbitrary, for any $k \in \mathbb{N}$ and any $a_k \in \mathbb{R}^+, a_{k-1} \in \mathbb{R}^+, ..., a_1 \in \mathbb{R}^+, a_0 \in \mathbb{R}^+$,
$a_k n^k + a_{k-1} n^{k-1} + ... + a_1 n + a_0 \in \Theta(n^k)$.

11. Prove that for any $k \in \mathbb{N}$, $1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1})$.

Solution:
Let $k \in \mathbb{N}$.

Let $c_1 = 1/2^{k+1}$, $c_2 = 1$. Then, $c_1 \in \mathbb{R}^+$ and $c_2 \in \mathbb{R}^+$. Let $B = 1$. Then, $B \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Suppose $n \geq 1$.

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}.$$

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n+1}{2}}\rceil^n i^k \geq \sum_{i=\lceil \frac{n+1}{2}}^{n} \lceil \frac{n}{2} \rceil^k = \lceil \frac{n}{2} \rceil \cdot \lceil \frac{n}{2} \rceil^k = \lceil \frac{n}{2} \rceil^{k+1} \geq \frac{n^{k+1}}{2^{k+1}}.$$

Hence, $n \geq 1 \to n^{k+1}/2^{k+1} \leq \sum_{i=1}^{n} i^k \leq n^{k+1}$,
and since n is an arbitrary real number, $\forall n \in \mathbb{N}, n \geq 1 \to n^{k+1}/2^{k+1} \leq \sum_{i=1}^{n} i^k \leq n^{k+1}$.

Therefore, $\exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \to c_1 \cdot n^{k+1} \leq \sum_{i=1}^{n} i^k \leq c_2 \cdot n^{k+1}$.

Since k is an arbitrary real number, for any $k \in \mathbb{N}$, $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$.

12. Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and let

$$O(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geq 0} | \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \to g(n) \leq cf(n)\}$$

Prove or disprove (in structured proof form) the following claims:

(a) Suppose $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Then $g \in O(f) \Rightarrow g(n^3) \notin O(f)$

Sample solution: The claim is false. I prove the negation. Let $F = \{h : \mathbb{N} \to \mathbb{R}^{\geq 0}\}$.

$$\exists f \in F, \exists g \in F, g \in O(f) \wedge g(n^3) \in O(f).$$

Let $f(n) = g(n) = 1$ for $n \in \mathbb{N}$. (constant functions).

Then $f, g \in F$, since $1 \in \mathbb{R}^+$. Let $c = 1$ and let $B = 0$.

Then $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.
Let $n \in \mathbb{N}$. Assume $n \geq B = 0$.

Then $g(n) = 1 \leq cf(n) = 1$. (by choice of f, g, and c).
Also $g(n^3) = 1 \leq cf(n)$. (by choice of f, g, and c).

So $n \geq B \Rightarrow g(n) \leq cf(n)$.
Also $n \geq B \Rightarrow g(n^3) \leq cf(n)$
Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
Also, since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow g(n^3) \leq cf(n)$.

Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
So $g \in O(f)$. (definition of $O(f)$).
Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n^3) \leq cf(n)$.
So $g(n^3) \in O(f)$. (definition of $O(f)$).

Since f and g are in F, $\exists f \in F, \exists G \in F, g \in O(f) \wedge g(n^3) \in O(f)$.

(b) Suppose $f, g, h : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Then $(g \in O(f) \wedge h \in O(f)) \Rightarrow \max(g, h) \in O(f)$.

Sample solution: The statement is true. Define (for convenience) $F = \{h : \mathbb{N} \to \mathbb{R}^+\}$.
Let $f, g \in F$. Assume $g \in O(f) \wedge h \in O(f)$.

Then $g \in O(f)$.
So $\exists c \in \mathbb{R}^+, B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$. (definition of $g \in O(f)$.).
Let $c_1 \in \mathbb{R}^+$ and $B_1 \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_1 \Rightarrow g(n) \leq c_1 f(n)$.
Then $h \in O(f)$.
So $\exists c \in \mathbb{R}^+, B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow h(n) \leq cf(n)$. (definition of $g \in O(f)$).
Let $c_2 \in \mathbb{R}^+$ and $B_2 \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_2 \Rightarrow h(n) \leq c_2 f(n)$.
Let $c = c_1 + c_2$. Let $B = \max(B_1, B_2)$.

Then $c \in \mathbb{R}^+$. (sum of positive real numbers is a positive real number).
Then $B \in \mathbb{N}$. (sum of natural numbers is a natural number).
Let $n \in \mathbb{N}$. Assume $n \geq B$.

Then $n \geq B_1$. (Since $B = \max(B_1, B_2)$).
So $g(n) \leq c_1 f(n)$. (shown above).
So $g(n) \leq (c_1 + c_2) f(n)$. (since $c_2 f(n)$ is non-negative).
Then $n \geq B_2$. (since $B = \max(B_1, B_2)$).
So $h(n) \leq c_2 f(n)$. (shown above).
So $h(n) \leq (c_1 + c_2) f(n)$ (since $c_1 f(n)$ is non-negative).
So $\max(g(n), h(n)) \leq (c_1 + c_2) f(n) = cf(n)$. (by the definition of max and choice of c).

Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow \max(g(n), h(n)) \leq cf(n)$.

Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow \max(g(n), h(n)) \leq cf(n)$.
So $\max(g, h) \in O(f)$. (definition of $O(f)$).
Thus $(g \in O(f) \wedge h \in O(f)) \Rightarrow \max(g, h) \in O(f)$.

Since f, g, and h are arbitrary elements of F, $\forall f \in F, \forall g \in F, \forall h \in F, (g \in O(f) \wedge h \in O(f)) \Rightarrow \max(g, h) \in O(f)$.

(c) Suppose $f, f', g, g' : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and $f \circ g(n) = f(g(n))$, $f' \circ g'(n) = f'(g'(n))$. Then $(f \in O(f') \wedge g \in O(g')) \Rightarrow f \circ g \in O(f' \circ g')$.

Sample solution: The claim is false. Let $F = \{h : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$. I will prove the negation:

$$\exists f \in F, \exists f' \in F, \exists g \in F, \exists g' \in F, f \in O(f') \wedge g \in O(g') \wedge f \circ g \notin O(f' \circ g')$$

Sample solution: The statement is false. I prove the negation (where $F = \{h : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$):

$$\exists f, f', g, g' \in F, f \in O(f') \wedge g \in O(g') \wedge f \circ g \notin O(f' \circ g')$$

Let $f(n) = 2^n = f'(n)$. Let $g(n) = 2n$. Let $g'(n) = n$.

Then $f, f', g, g' \in F$. (Their range if $\mathbb{N} \subset \mathbb{R}^{\geq 0}$).
Let $c_1 = 1$. Let $B_1 = 0$.

Then $c_1 \in \mathbb{R}^+$ and $B_1 \in \mathbb{N}$.
Let $n \in \mathbb{N}$. Assume $n \geq B_1 = 0$.

Then $f(n) = 2^n \leq c_1 f'(n) = 2^n$. (by choice of f, f', and c).

So $n \geq B_1 \Rightarrow f(n) \leq cf'(n)$.
Since n is an arbitrary element of $\mathbb{N}$, $n \geq B_1 \Rightarrow f(n) \leq cf'(n)$.

Since $c_1 \in \mathbb{R}^+$ and $B_1 \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cf'(n)$.
So $f \in O(f')$. (definition).
Let $c_2 = 2$. Let $B_2 = 0$.

Then $c_2 \in \mathbb{R}^+$ and $B_2 \in \mathbb{N}$.
Let $n \in \mathbb{N}$. Assume $n \geq B_2$.

Then $g(n) = 2n \leq 2n = c_2 g'(n)$. (by choice of g, g', and c_2).

So $n \geq B_2 \Rightarrow g(n) \leq c_2 g'(n)$.
Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B_2 \Rightarrow g(n) \leq c_2 g'(n)$.

Since $c_2 \in \mathbb{R}^+$ and $B_2 \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \to g(n) \leq cg'(n)$.
So $g \in O(g')$. (definition).
Let $c \in \mathbb{R}^+$. Let $B \in \mathbb{N}$.

Let $n = \max(\lceil[\log_2 c] + 1\rceil, B)$.

Then $n \in \mathbb{N}$. (maximum of two natural numbers).
Then $n \geq B$. (definition of $\max$).
Then $2^n \geq 2^{(\log_2 c)+1} = 2c > c$. (by choice of n).
So $2^n \times 2^n > c2^n$. (multiply the inequality by 2^n).
So $f(g(n)) = f(2n) = 2^{2n} = 2n \times 2n > c2n = c2^{g'(n)} = f'(g'(n))$. (by the previous inequality and choice of f, f', g, g').
So $n \geq B$ and $f(g(n)) > f'(g'(n))$.
Since $n \in \mathbb{N}$, $\exists n \in \mathbb{N}, n \geq B \wedge f(g(n)) > f'(g'(n))$.
Since c is an arbitrary element of $\mathbb{R}^+$ and B is an arbitrary element of $\mathbb{N}$, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge f(g(n)) > f'(g'(n))$
So $f \circ g \notin O(f' \circ g')$. (definition of $O(f'g')$).
Since f, f', g, g' are elements of $\mathbb{F}$, $\exists f, f', g, g' \in F, f \in O(f') \wedge g \in O(g') \wedge f \circ g \notin O(f' \circ g')$

(d) Suppose $f, g, h : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and $gh(n) = g(n)h(n)$. Then $(g \in O(f) \wedge h \in O(f)) \to gh \in O(f)$.
Sample solution: The statement is false. Let $F = \{h : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$. I will prove the negation:

$$\exists f, g, h \in F, g \in O(f) \wedge h \in O(f) \wedge gh \notin O(f)$$

Let $f(n) = g(n) = h(n) = n$.
Then $f, g, h \in F$. (since they map $\mathbb{N}$ to $\mathbb{N} \subset \mathbb{R}^{\geq 0}$).
Let $c = 1$. Let $B = 0$.
Then $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.
Let $n \in \mathbb{N}$. Assume $n \geq B = 0$.
Then $g(n) = n \leq n = cf(n)$. (by the choice of f, g, and c).
Then $h(n) = n \leq n = cf(n)$. (by the choice of f, h, and c).
So $n \geq B \Rightarrow g(n) \leq cf(n)$.
Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
So $n \geq B \Rightarrow h(n) \leq cf(n)$.
Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow h(n) \leq cf(n)$.
Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
So $g \in O(f)$. (definition)
Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}R^+, B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow h(n) \leq cf(n)$.
So $h \in O(f)$. (definition).
Let $c \in \mathbb{R}^+$. Let $B \in \mathbb{N}$.
Let $n = \max(B, \lceil c \rceil + 1)$.
Then $n \in \mathbb{N}$. (maximum of natural numbers, since c is positive, its ceiling is a natural number, and the sum of natural numbers is a natural number).
Then $n \geq B$ (definition of max).
Then $g(n)h(n) = n^2 = n \times n > cn$ (since $n \geq c + 1 > c$).
So $g(n)h(n) > cf(n)$.
Since $n \in \mathbb{N}$, $\exists n \in \mathbb{N}, n \geq B \wedge g(n)h(n) > cf(n)$.
Since c is an arbitrary element of $\mathbb{R}^+$ and B is an arbitrary element of $\mathbb{N}$, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge g(n)h(n) > cf(n)$.
So $g, h \notin O(f)$. (definition of $O(f)$, negated).
Since $f, g, h \in F$, $\exists f, g, h \in F, g \in O(f) \wedge h \in O(f) \wedge gh \notin O(f)$.

(e) Suppose $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and $f'(n) = \lfloor f(n) \rfloor, g'(n) = \lfloor g(n) \rfloor$. Then $g \in O(f) \to g' \in O(f')$.
Sample solution: The statement is false. Let $F = \{h : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$. I prove the negation:

$$\exists f, g, f', g' \in F, f'(n) = \lfloor f(n) \rfloor, g'(n) = \lfloor g(n) \rfloor \wedge g \in O(f) \wedge g' \notin O(f')$$

Let $f(n) = 0.5$, $f'(n) = \lfloor f(n) \rfloor = 0$, $g(n) = 1$, $g'(n) = \lfloor g(n) \rfloor = 1$. (constant functions).

Then $f, f', g, g' \in F$ (they map natural numbers to non-negative real numbers).
Let $c = 2$. Let $B = 0$.

Let $n \in \mathbb{N}$. Assume $n \geq B = 0$.

Then $g(n) = 1 \leq 1 = cf(n)$. (by choice of f, g, c).

So $n \geq B \Rightarrow g(n) \leq cf(n)$.
Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq f(n)$.

Since $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
So $g \in O(f)$. (definition).
Let $c' \in \mathbb{R}^+$. Let $B' \in \mathbb{N}$

Let $n = B'$.

Then $n \in \mathbb{N}$ (since B' is).
Also $n \geq B'$ (since $n = B'$).
Also $1/c' > 0$ (since reciprocal of a positive number is positive).
So $1 > c'0$. (multiplying both sides by positive c').
So $g'(n) = 1 > c'0 = c'f'(n)$.

Since $n \in \mathbb{N}$, $\exists n \in \mathbb{N}, n \geq B' \wedge g'(n) > c'f'(n)$.

Since c' is an arbitrary element of $\mathbb{R}^+$ and B' is an arbitrary element of $\mathbb{N}$, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge g'(n) > cf'(n)$.
So $g' \notin O(f')$. (definition of $O(f')$, negated).

Since $f, f', g, g' \in F$, $\exists f, f', g, g' \in F, f'(n) = \lfloor f(n) \rfloor \wedge g'(n) = \lfloor g(n) \rfloor \wedge g \in O(f) \wedge g' \notin O(f')$.

Floating Point Systems

1. Recall that the binary number $(b_{k-1}b_{k-2}\dots b_1b_0)_2$ has value $b_{k-1}2^{k-1} + b_{k-2}2^{k-2} + \cdots + b_1 2^1 + b_0 2^0$.

   (a) What decimal numbers are represented by the binary numbers $(001010)_2$, $(010100)_2$, $(011110)_2$, $(101000)_2$, $(110010)_2$? Show your computations.

   SOLN:

$$\begin{aligned}
(001010)_2 &= 8+2=10 \\
(010100)_2 &= 16+4=20 \\
(011110)_2 &= 16+8+4+2=30 \\
(101000)_2 &= 32+8=40 \\
(110010)_2 &= 32+16+2=50
\end{aligned}$$

   (b) Give binary representations for the decimal numbers 10, 42, 170, 682. Show your computations. **SOLN:**

$$\begin{aligned}
10 &= 8+2=(1010)_2 \\
42 &= 32+8+2=(101010)_2 \\
170 &= 128+32+8+2=(10101010)_2 \\
682 &= 512+128+32+8+2=(1010101010)_2
\end{aligned}$$

   (c) Multiply the two binary numbers $(10110)_2$ and $(101)_2$, **in binary**, *i.e.*, carry out all of the operations (including addition) in binary notation, *without* converting to decimal notation. Show your computation. **SOLN:**

   $(10110)_2 * (101)_2 = (1101110)_2$:

```
   10110
  *00101
 -------
   10110
 1011000
 -------
 1101110
```

   (d) If $(b_{k-1}b_{k-2}\dots b_1b_0)_2$ is a binary representation for natural number x (where each b_i is either 0 or 1), give a binary representation of the number $4x+3$. **SOLN:**

   If

$$\begin{aligned}
x &= (b_{k-1}b_{k-2}...b_1b_0)_2, \text{ then} \\
4x &= (b_{k-1}b_{k-2}...b_1b_0 00)_2 \\
4x+3 &= (b_{k-1}b_{k-2}...b_1b_0 11)_2.
\end{aligned}$$

2. Consider a normalized floating point system with base $\beta = 10$, $t = 3$, $e_{min} = -4$ and $e_{max} = +2$.

   (a) Evaluate this polynomial, $3.12x^3 - 2.11x^2 + 4.01x + 10.2$, with $x = 1.32$. Proceed from left to right, using round to nearest. Show your work. **Soln:**
   x is represented as 1.32×10^0
   $x^2 = 1.32 \cdot 1.32 = 1.7424$ is represented as 1.74×10^0
   $x^3 = 1.32 \cdot 1.74 = 2.2968$ is represented as 2.30×10^0
   $3.12 \cdot x^3 = 3.12 \cdot 2.3 = 7.176$ is represented as 7.18×10^0
   $-2.11 \cdot x^2 = -2.11 \cdot 1.74 = -3.6714$ is represented as -3.67×10^0
   $4.01 \cdot x = 4.01 \cdot 1.32 = 5.2932$ is represented as 5.29×10^0
   10.2 is represented as 1.02×10^1

   $3.12 \cdot x^3 - 2.11 \cdot x^2 = 7.18 - 3.67 = 3.51$ is represented as 3.51×10^0
   $3.12 \cdot x^3 - 2.11 \cdot x^2 + 4.01 \cdot x = 3.51 + 5.29 = 8.8$ is represented as 8.8×10^0
   $3.12 \cdot x^3 - 2.11 \cdot x^2 - 4.01 \cdot x + 10.2 = 8.8 + 10.2 = 19$ is represented as 1.9×10^1

   (b) Evaluate the polynomial proceeding from right to left, using round to nearest. Show your work. **Soln:**
   $4.01 \cdot x + 10.2 = 5.29 + 10.2 = 15.49$ is represented as 1.55×10^1
   $-2.11 \cdot x^2 + 4.01 \cdot x + 10.2 = -3.67 + 1.55 \times 10^1 = 11.83$ is represented as 1.18×10^1
   $3.12 \cdot x^3 - 2.11 \cdot x^2 + 4.01 \cdot x + 10.2 = 7.18 + 1.18 \times 10^1 = 18.98$ is represented as 1.90×10^1

   (c) Which calculation is more stable? Explain your answer. **Soln:**
   For this value of x, both calculations have the same result. Therefore, the result has the same relative error, and the calculations are equally stable.

3. Consider a normalized floating point system with base $\beta = 3$, $t = 3$, $e_{min} = -3$ and $e_{max} = +2$.

   (a) What is the smallest (base 10) number that can be represented in this system? Show your work. **Soln:**
   The representation in this floating point system of the smallest number is 1.00×3^{-3} which in base 10 is $1/27$.

   (b) What is the largest (base 10) number that can be represented in this system? Show your work. **Soln:**
   The representation of the largest number in this floating point system is 2.22×3^2 which in base 3 is 222 and in base 10 is $2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0 = 18 + 6 + 2 = 26$.

   (c) Give examples of real (base 10) numbers, that will cause overflow and underflow when represented in this floating point system. Show your work. **Soln:**
   In this system the smallest positive number is represented as 1.00×3^{-3}. Hence any number positive number between 0 and $1/27$ cannot be represented. So, an example of underflow is the number $1/28$.

   Overflow occurs at the representation of any number larger than 26. So, an example of overflow is the number 27.

4. Consider the base -2 representation, where $b_i \in \{0, 1\}$, and

$$(b_n b_{n-1} \cdots b_1 b_0)_{-2} = \sum_{i=0}^{n} b_i (-2)^i.$$

   (a) How many positive integers can be written in an n-digit base -2 representation (including representations with leading zeros on the left)? How many negative numbers can be written in an n-digit base -2 representation (including representations with leading zeros on the left). Justify your answer.

   Sample solution: You have at your disposal bits $(b_{n-1} \cdots b_0)_{-2} = \sum_{i=0}^{n-1} b_i (-2)^i$, where $b_i \in \{0, 1\}$. There are two cases to consider, depending on whether n is even or odd:

Case 1, n is odd: In this case $n-1$ is even. There are 2^0 positive base -2 numbers of the form $(0\cdots 01)_{-2}$. In general there are 2^{2k} positive base -2 numbers of the form $(0\cdots 01b_{2k-1}\cdots b_0)_{-2}$, where $0 \leq 2k \leq n-1$. We know these numbers are positive because $1 \times (-2)^{2k}$ dominates all the other terms. Summing this geometric series gives us:

$$\sum_{k=0}^{(n-1)/2} 2^{2k} = \sum_{k=0}^{(n-1)/2} 4^k = \frac{4^{(n+1)/2}-1}{3}$$

…positive numbers.
Similarly, there are 2^1 negative base -2 numbers of the form $(0\cdots 01b_0)_{-2}$. In general there are 2^{2k+1} negative base -2 numbers of the form $(0\cdots 01b_{2k}\cdots b_0)_{-2}$, where $0 \leq 2k \leq n-3$. Summing this geometric series give us

$$\sum_{k=0}^{(n-3)/2} 2^{2k+1} = 2\sum_{k=0}^{(n-3)/2} 4^k = 2\frac{4^{(n-1)/2}-1}{3}$$

…negative numbers.

Case 2, n is even: In this case $n-1$ is odd, so we must sum the geometric series with different upper bounds. There are 2^{2k+1} negative base -2 numbers of the form $(0\cdots 1b_{2k}\cdots b_0)_{-2}$, where $0 \leq 2k \leq n-2$, so we sum the geometric series to get

$$\sum_{k=0}^{(n-2)/2} 2^{2k+1} = 2\sum_{k=0}^{(n-2)/2} 4^k = 2\frac{4^{n/2}-1}{3}$$

…negative numbers.
There are 2^{2k} positive base -2 numbers of the form $(0\cdots 01b_{2k-1}\cdots b_0)_{-2}$, where $0 \leq 2k \leq n-2$, so summing the geometric series gives us

$$\sum_{k=0}^{(n-2)/2} 2^{2k} = \sum_{k=0}^{(n-2)/2} 4^k = \frac{4^{n/2}-1}{3}$$

(b) For natural number k, what is the minimum number of digits needed to represent k in base -2 representation? Justify your answer.

Sample solution: First observe that all the base -2 numbers with an even number of digits are negative, so we'll need k to be represented with $2j+1$ digits, for some natural number j. The largest number that can be represented with $2j+1$ digits is $(101\cdots 01\cdots 01)_{-2}$, which is $\sum_{i=0}^{j}(-2)^{2j} = \sum_{i=0}^{j} 4^j$ $= (4^{j+1}-1)/3$. The smallest number (still positive, since it is dominated by $1 \times (-2)^{2j}$) that can be represented with $2j+1$ digits is $(110\cdots 10\cdots 10)_{-2} = (-2)^{2j} + \sum_{i=0}^{j-1}(-2)^{2i+1} = 4^j - 2\sum_{i=0}^{j-1} 4^i$ $= 4^j - 2(4^j-1)/3 = (4^j+2)/3$. This means that:

$$\begin{aligned}
\frac{4^{j+1}-1}{3} &\geq k \geq \frac{4^j+2}{3} \\
\Rightarrow 4^{j+1}-1 &\geq 3k \geq 4^j+2 \\
\Rightarrow \lfloor \log_4(4^{j+1}-1) \rfloor &\geq \lfloor \log_4(3k) \rfloor \geq \lfloor \log_4(4^j+2) \rfloor \\
j &\geq \lfloor \log_4 3k \rfloor \geq j
\end{aligned}$$

Thus k requires $2\lfloor \log_4 3k \rfloor + 1$ base -2 digits.

5. How many non-zero numbers are there in the normalized floating-point system with $\beta = 4$, $t = 4$, $e_{min} = -4$, $e_{max} = 4$? Show your work.

Solution: There are:

- 2 possibilities for the sign ($\pm$),
- 3 possibilities for the first digit $(1, 2, 3)$, since it must be different from 0,
- 4 possibilities for each of the other three digits,
- 9 possibilities for the exponent (-4–4),

for a total of $2 \times 3 \times 4 \times 4 \times 4 \times 9 = 3456$ non-zero numbers.

6. (a) Does the addition of numbers exactly representable in a floating-point system always produce numbers exactly representable in the system? Justify.

   Solution: No.

   Consider the floating-point system with $\beta = 10$, $t = 3$, $e_{\min} = -2$, $e_{\max} = +2$. Let $x = 500 = 5.00 \times 10^2$ and $y = 500 = 5.00 \times 10^2$. Then, x and y are numbers exactly representable in the system but $x + y = 500 + 500 = 1000$ is not exactly representable because of overflow.

   Alternatively, if $x = 100 = 1.00 \times 10^2$ and $y = 0.01 = 1.00 \times 10^{-2}$, then $x + y = 100.01$ is not exactly representable in the system even though it does not cause overflow.

   (b) Does the subtraction of numbers exactly representable in a floating-point system always produce numbers exactly representable in the system? Justify.

   Solution: No.

   Consider the floating-point system with $\beta = 10$, $t = 3$, $e_{\min} = -2$, $e_{\max} = +2$. Let $x = -500 = -5.00 \times 10^2$ and $y = 500 = 5.00 \times 10^2$. Then, x and y are numbers exactly representable in the system but $x - y = -500 - 500 = -1000$ is not exactly representable because of overflow.

   Alternatively, if $x = 100 = 1.00 \times 10^2$ and $y = 0.01 = 1.00 \times 10^{-2}$, then $x - y = 99.99$ is not exactly representable in the system even though it does not cause overflow.

   (c) Does the multiplication of numbers exactly representable in a floating-point system always produce numbers exactly representable in the system? Justify.

   Solution: No.

   Consider the floating-point system with $\beta = 10$, $t = 3$, $e_{\min} = -2$, $e_{\max} = +2$. Let $x = 2 = 2.00 \times 10^0$ and $y = 500 = 5.00 \times 10^2$. Then, x and y are numbers exactly representable in the system but $x \cdot y = 2 \cdot 500 = 1000$ is not exactly representable because of overflow.

   Alternatively, if $x = 0.01 = 1.00 \times 10^{-2}$ and $y = 0.01 = 1.00 \times 10^{-2}$, then $x \cdot y = 0.0001$ is not exactly representable in the system because of underflow.

   Alternatively, if $x = 101 = 1.01 \times 10^2$ and $y = 0.101 = 1.01 \times 10^{-1}$, then $x \cdot y = 10.201$ is not exactly representable in the system even though it does not cause overflow or underflow.

   (d) Does the division of numbers exactly representable in a floating-point system always produce numbers exactly representable in the system? Justify.

   Solution: No.

   Consider the floating-point system with $\beta = 10$, $t = 3$, $e_{\min} = -2$, $e_{\max} = +2$. Let $x = 500 = 5.00 \times 10^2$ and $y = 0.1 = 1.00 \times 10^{-1}$. Then, x and y are numbers exactly representable in the system but $x/y = 500/0.1 = 5000$ is not exactly representable because of overflow.

   Alternatively, if $x = 0.1 = 1.00 \times 10^{-1}$ and $y = 100 = 1.00 \times 10^2$, then $x/y = 0.001$ is not exactly representable in the system because of underflow.

   Alternatively, if $x = 1 = 1.00 \times 10^0$ and $y = 3 = 3.00 \times 10^0$, then $x/y = 0.333\ldots$ is not exactly representable in the system even though it does not cause overflow or underflow.

7. In lecture, we calculated an upper bound of $\beta^{-p+1}/2$ for the relative error introduced by rounding to nearest in a floating-point system. Compute an upper bound for "round towards zero" (*i.e.*, round down for positive numbers, round up for negative numbers).

   Solution:

Case 1: If $x > 0$, then let

$$x = +d_0.d_1d_2 \dots d_{t-1}d_t \cdots \times \beta^e$$

where $d_0 \neq 0$ and $d_i \neq 0$ for some $i \geq t$ (otherwise x can be represented exactly). Then, x is rounded to

$$d_0.d_1d_2 \dots d_{t-1} \times \beta^e$$

with an absolute error of

$$0.00 \dots 0d_t \cdots \times \beta^e \leq 1.00 \dots 0 \times \beta^{e-t+1}.$$

Since $x \geq 1.00 \dots 0 \times \beta^e$, the relative error is therefore at most

$$\frac{\beta^{e-t+1}}{|x|} \leq \frac{\beta^{e-t+1}}{1.0 \times \beta^e} = \beta^{-t+1}.$$

Case 2: If $x < 0$, then the same argument as for $x > 0$ applies: let

$$x = -d_0.d_1d_2 \dots d_{t-1}d_t \cdots \times \beta^e$$

where $d_0 \neq 0$ and $d_i \neq 0$ for some $i \geq t$ (otherwise x can be represented exactly). Then, x is rounded to

$$-d_0.d_1d_2 \dots d_{t-1} \times \beta^e$$

with an absolute error of

$$0.00 \dots 0d_t \cdots \times \beta^e \leq 1.00 \dots 0 \times \beta^{e-t+1}.$$

Since $x \leq -1.00 \dots 0 \times \beta^e$, the relative error is therefore at most

$$\frac{\beta^{e-t+1}}{|x|} \leq \frac{\beta^{e-t+1}}{1.0 \times \beta^e} = \beta^{-t+1}.$$

In all cases, the relative error in "round-to-zero" is always no more than β^{1-t}.

8. Consider evaluating the function $f(x) = +\sqrt{1+x} - \sqrt{x}$. Is it stable to calculate it in Java with the expression `Math.sqrt(x+1)-Math.sqrt(x)`? Can you find a more stable approach? Justify.

Solution: The computation `Math.sqrt(x+1)-Math.sqrt(x)` is not stable because $+\sqrt{1+x}$ will generally be close to $-\sqrt{x}$ in magnitude, so there will be catastrophic cancellation.

However,

$$(+\sqrt{1+x} - \sqrt{x})\frac{+\sqrt{1+x} + \sqrt{x}}{+\sqrt{1+x} + \sqrt{x}} = \frac{(1+x) - x}{+\sqrt{1+x} + \sqrt{x}} = \frac{1}{+\sqrt{1+x} + \sqrt{x}}$$

so the Java expression `1.0 / (Math.sqrt(x+1) + Math.sqrt(x))` computes the same value in a stable manner.