Tutorial 10/26/09 — sample solution

$$\text{S1:} \quad \forall n_1 \in \mathbb{N}, \exists n_2 \in \mathbb{N}, \forall n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$$

1. Write the proof structure for the direct proof of S1.

Assume $n_1 \in \mathbb{N}$. # generic element of $\mathbb{N}$.
  Choose $n_2 =$???. # choice may depend on n_1
  $\vdots$ # may have to prove n_2 is natural
  Then $n_2 \in \mathbb{N}$. # instantiated existential
  Assume $n_3 \in \mathbb{N}$. # generic element of $\mathbb{N}$
    Assume $P(n_1, n_2, n_3)$. # assume the antecedent
      $\vdots$# some arguments leading to Q
      Then $Q(n_1, n_2, n_3)$. # derive the consequent
    Then $P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # antecedent leads to consequent
  Then $\forall n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # n_3 was generic
  Then $\exists n_2 \in \mathbb{N}, \forall n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # chose n_2 that worked
Conclude $\forall n_1 \in \mathbb{N}, \exists n_2 \in \mathbb{N}, \forall n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # n_1 was generic
Conclude S1. # previous line

2. Write the structure for the direct disproof of S1.

Choose $n_1 =$???. # crafty choice to refute the claim
$\vdots$ # may have to prove n_1 is natural
Then $n_1 \in \mathbb{N}$. # instantiated existential
Assume $n_2 \in \mathbb{N}$. # generic natural number
  Choose $n_3 =$???. # another crafty choice, may depend on n_2
  $\vdots$ # may have to prove n_3 is natural
  Then $n_3 \in \mathbb{N}$. # instantiated existential
  $\vdots$ # arguments leading to $P(n_1, n_2, n_3)$
  Then $P(n_1, n_2, n_3)$. # antecedent follows
  $\vdots$ # arguments leading to $\neg Q(n_1, n_2, n_3)$
  Then $\neg Q(n_1, n_2, n_3)$. # negation of consequent follows
  Then $P(n_1, n_2, n_3) \wedge \neg Q(n_1, n_2, n_3)$. # conjunction
  $\exists n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \wedge \neg Q(n_1, n_2, n_3)$. # chose n_3 that works
Then $\forall n_2 \in \mathbb{N}, \exists n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \wedge \neg Q(n_1, n_2, n_3)$. # n_2 was generic
Conclude $\exists n_1 \in \mathbb{N}, \forall n_2 \in \mathbb{N}, \exists n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \wedge \neg Q(n_1, n_2, n_3)$. # chose n_1 that works
Conclude $\neg$S1. # previous line

3. Write the structure for proof by contradiction of S1.

Assume $\neg$S1. # to derive a contradiction
  Then $\exists n_1 \in \mathbb{N}, \forall n_2 \in \mathbb{N}, \exists n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \wedge \neg Q(n_1, n_2, n_3)$. # by assumption
  $\vdots$ # some arguments that lead to contradiction
  contradiction! # yeehaw!
Conclude S1. # since assuming $\neg$S1 led to contradiction

4. Write the proof structure for the proof of the contrapositive of S1.

Assume $n_1 \in \mathbb{N}$. # generic element of $\mathbb{N}$.
  Choose $n_2 =???$. # choice may depend on n_1
  $\vdots$ # may have to prove n_2 is natural
  Then $n_2 \in \mathbb{N}$. # instantiated existential
  Assume $n_3 \in \mathbb{N}$. # generic element of $\mathbb{N}$
    Assume $\neg Q(n_1, n_2, n_3)$. # assume the negation of the consequent
      $\vdots$# some arguments leading to $\neg P$
      Then $\neg P(n_1, n_2, n_3)$. # derive the negation of the antecedent
    Then $\neg Q(n_1, n_2, n_3) \Rightarrow \neg P(n_1, n_2, n_3)$. # $\neg Q$ leads to $\neg P$
  Then $\forall n_3 \in \mathbb{N}, \neg Q(n_1, n_2, n_3) \Rightarrow \neg P(n_1, n_2, n_3)$. # n_3 was generic
  Then $\exists n_2 \in \mathbb{N}, \forall n_3 \in \mathbb{N}, \neg Q(n_1, n_2, n_3) \Rightarrow \neg P(n_1, n_2, n_3)$. # chose n_2 that worked
Conclude $\forall n_1 \in \mathbb{N}, \exists n_2 \in \mathbb{N}, \forall n_3 \in \mathbb{N}, \neg Q(n_1, n_2, n_3) \Rightarrow \neg P(n_1, n_2, n_3)$. # n_1 was generic
Conclude $\forall n_1 \in \mathbb{N}, \exists n_2 \in \mathbb{N}, \forall n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # by proof of contrapositive
Conclude S1. # previous line

5. Write the proof structure for the direct proof of S1, assuming you need separate cases when n_1 is even or odd.

Assume $n_1 \in \mathbb{N}$. # generic element of $\mathbb{N}$.
  Case 1, assume n_1 even:
    Choose $n_2 =???$. # choice may depend on even n_1
    $\vdots$ # may have to prove n_2 is natural
    Then $n_2 \in \mathbb{N}$. # instantiated existential
    Assume $n_3 \in \mathbb{N}$. # generic element of $\mathbb{N}$
      Assume $P(n_1, n_2, n_3)$. # assume the antecedent
        $\vdots$# some arguments leading to Q
        Then $Q(n_1, n_2, n_3)$. # derive the consequent
      Then $P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # antecedent leads to consequent
    Then $\forall n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # n_3 was generic
    Then $\exists n_2 \in \mathbb{N}, \forall n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # chose n_2 that worked
  Case 2, assume n_1 odd:
    Choose $n_2 =???$. # may depend on odd n_1
    $\vdots$ # may need to prove n_2 is natural
    Then $n_2 \in \mathbb{N}$. # instantiated existential
    Assume $n_3 \in \mathbb{N}$. # generic element of $\mathbb{N}$
      Assume $P(n_1, n_2, n_3)$. # assume the antecedent
        $\vdots$# some arguments leading to Q
        Then $Q(n_1, n_2, n_3)$. # derive the consequent
      Then $P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # antecedent leads to consequent
    Then $\forall n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # n_3 was generic
    Then $\exists n_2 \in \mathbb{N}, \forall n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # chose n_2 that worked
Then $\forall n_1 \in \mathbb{N}, \exists n_2 \in \mathbb{N}, \forall n_3 \in \mathbb{N}, P(n_1, n_2, n_3) \Rightarrow Q(n_1, n_2, n_3)$. # worked in both possible cases