CSC 165

bounds

week 9, lecture 1

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Wacky Wed — what about tutorials, stay tuned –

A2 – getting closer ...

$f(c)$ st $10^{f(c)} = c$

$\forall n \in \mathbb{N},$

We've proved: $P(n) : 2^n \geq 2n$

$n^2, n^3, n^k \log_2 n$

Use this to prove that $2^n \notin \mathcal{O}(n)$

$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > cn$

assume $c \in \mathbb{R}^+$, assume $B \in \mathbb{N}$

Choose $n = 2(\lceil \log c \rceil + B + 1)$ Then $n \in \mathbb{N}$

$\lceil \log c \rceil \in \mathbb{N}$, $B, 1, \in \mathbb{N}$, $\mathbb{N}$ closed under $+$

Then $n \geq B$

Then $2^n = 2^{n/2} 2^{n/2}$

$\geq 2^{n/2} \cdot n$ # $P(n/2)$, $n/2 \in \mathbb{N}$

$> cn$

$n/2 > \log c$

by choice of n

2^x monotonic (strictly)

$\lim_{x \to \infty} \frac{2^x}{x}$

$2^{\log x} \equiv x$ (for pos x)

scratch

bounded below

Notice that the definition of big-Omega differs in just one character from big-Oh:

$\Omega(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \geq cg(n)\}$

The rôle of B is, as with big-Oh, to act as a breakpoint, so comparisons don't have to start at the origin

The rôle of c is to scale g *down* below f.

If you're proving $f \in \Omega(g)$, you get to choose c and B to suit your proof.

one last bound

It often happens that functions are bounded above *and* below by the same function. In other words, $f \in \mathcal{O}(G) \wedge f \in \Omega(g)$. We combine these two concepts into $f \in \Theta(g)$.

$$\Theta(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

You might want to draw pictures, and conjecture about values of c_1, c_2, B for $f = 5n^2 + 15$ and $g = n^2$.

$\mathcal{F} = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0}\}$

some theorems

$\forall f, g, h \in \mathcal{F},$

How do you deal with a general statement about two functions:

$(f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)) \Rightarrow f \in \mathcal{O}(h)$

Assume $f, g, h \in \mathcal{F}$

Assume $f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)$ # antecedent

Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c\,g(n)$

Choose $c_1 \in \mathbb{R}, B_1 \in \mathbb{N}$, s.t. $\forall n \in \mathbb{N}, n \geq B_1 \Rightarrow f(n) \leq c_1 g(n)$ (*)

Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq c\,h(n)$

Choose $c_2 \in \mathbb{R}^+, B_2 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B_2 \Rightarrow g(n) \leq c_2 h(n)$ (**)

Choose $c' = c_1 c_2$. Then $c' \in \mathbb{R}^+$

Choose $B' = \max(B_1, B_2)$

Then $\forall n \in \mathbb{N}, n \geq B' \Rightarrow f(n) \leq c_1 \cdot g(n)$ # (*) $B' \geq B_1$

$\leq c_1 c_2 h(n)$ # (**) $B' \geq B_2$

Then $f \in \mathcal{O}(h)$ By

scratch

How about: $g \in \Omega(f) \Leftrightarrow f \in \mathcal{O}(g)$

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