CSC 165

polynomials

week 8, lecture 2

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pinning down intuition

We know, or have heard, that polynomials of the same degree grow at "roughly" the same speed

We want to make this "roughly" explicit

Here's how we define $\mathcal{O}(n^2)$, functions that eventually grow no more quickly than n^2

$$\mathcal{O}(n^2) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

The definition says that there's a multiplier, c, such that if you go far enough to the right, B, the graph of f is bounded above by the graph of cn^2

$3n^2+2n: \mathbb{N} \mapsto \mathbb{R}^{\geq 0}$ ✓ Prove $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2+2n \leq cn^2$

Prove: $3n^2 + 2n \in \mathcal{O}(n^2)$

Let $c = \underline{\ 5\ }$. Then $c \in \mathbb{R}^+$ #

Let $B = \underline{\ 0\ }$. Then $B \in \mathbb{N}$ #

Assume $n \in \mathbb{N}$ # generic choice

Assume $n \geq B$ # antecedent

Then $3n^2 + 2n \leq 3n^2 + 2n \cdot n$ # $n=0$ then $2n^2 = 2n$; $n \geq 1$ then $2n^2 \geq 2n$

$= 5n^2$

$= cn^2$ # $c = 5$

Then $n \geq B \Rightarrow 3n^2+2n \leq cn^2$ # introduced $\Rightarrow$

Then $\forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2+2n \leq cn^2$ # introduced $\forall$

Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2+2n \leq cn^2$

introduced $\exists$ twice

Conclude $3n^2+2n \in O(n^2)$ # matches definition

scratch

in general, $\mathcal{O}(g)$:

$$\mathcal{O}(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cg(n)\}$$

Prove: $7n^6 - 5n^4 + 2n^3 \in \mathcal{O}(6n^8 - 4n^5 + n^2)$

Let $c =$ ______. Then $c \in \mathbb{R}^+$ #

Let $B =$ ______. Then $B \in \mathbb{N}$ #

assume $n \in \mathbb{N}$ # generic choice

assume $n \geq B$ # antecedent

Then $7n^6 - 5n^4 + 2n^3 \leq 7n^6 + 2n^3$ # added $5n^4 \geq 0$

$\leq 7n^6 + 2n^6$ # mult by n^3 # won't decrease

$= 9n^6$

$\leq 9n^8$ # mult by n^2 won't # decrease

$= c\,2n^8$ # $c = 9/2$

$= c(6n^8 - 4n^8)$

$\leq c(6n^8 - 4n^5)$ # $4n^8 \geq 4n^5\ \forall n$

$\leq c(6n^8 - 4n^5 + n^2)$ # added $n^2 \geq 0$

Then $n \geq B \Rightarrow 7n^6 - 5n^4 + 2n^3 \leq c(6n^8 - 4n^5 + n^2)$ # intro $\Rightarrow$

~~scratch~~

Then

$\forall n \in \mathbb{N}, n \geq B \implies 7n^6 - 5n^4 + 2n^3 \leq c(6n^8 - 4n + n^2)$

intro $\forall$

Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 7n^6 - 5n^4 + 2n^3 \leq c(6n^8 - 4n + n^2)$

intro $\exists$ twice

<u>Conclude</u> $7n^6 - 5n^4 + 2n^3 \in O(6n^8 - 4n + n^2)$ # matches defn.

how to prove $n^4 \notin \mathcal{O}(3n^2)$

begin by stating the negation, then prove that

scratch