CSC 165

condition

week 12, lecture 2

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resources: chapter 7 of course notes

http://docs.python.org/tutorial/floatingpoint.html

diminishing errors

$ax^2 + \cancel{bx} + c = 0$

$ax^2 = -c$

We saw that the quadratic formula had two opportunities to experience catastrophic cancellation, both involving the parameter b in

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What's the situation when $b = 0$?

$$\frac{\sqrt{-4ac}}{2a}$$

This ends up being no worse than the square root operation, even if we have to deal with $i = \sqrt{-1}$. What does that do to input errors?

square root squashes error

Suppose we calculate $\sqrt{c}$ where the true value of $c = 0.25$,
but we calculate with a poor approximation $c' = 0.36$.

The relative error of the input is $|\overbrace{0.25 - 0.36}^{0.11}|/|0.25| = 0.11/0.25 = 44\%$
The relative error of the output is $|0.5 - 0.6|/|0.5| = 0.1/0.5 = 20\%$
Taking the square root halved the relative error!

This isn't a fluke due to some special choice of $c = 0.25$ and $c' = 0.36$.
Do the algebra to work out the general case, and square root always reduces relative error.

scratch

c - true
c' - approx

$$\frac{\text{re-out}}{\text{re-in}} \rightarrow \frac{\left(\dfrac{|\sqrt{c} - \sqrt{c'}|}{|\sqrt{c}|} \right)}{\left(\dfrac{|c-c'|}{|c|} \right)}$$

reciprocal

$$= \frac{|c|}{|c-c'|} \cdot \frac{|\sqrt{c} - \sqrt{c'}|}{\sqrt{c}} = \frac{|c|}{\sqrt{c}} \, \frac{|\sqrt{c} - \sqrt{c'}|}{|c-c'|}$$

$$|c-c'| = |(\sqrt{c} - \sqrt{c'})(\sqrt{c} + \sqrt{c'})|$$

$$= \frac{\sqrt{c}}{\sqrt{c} + \sqrt{c'}}$$

condition number

$$\frac{\text{re-in}}{\text{re-out}} : \frac{\dfrac{|f(x) - f(x')|}{|f(x)|}}{\left|\dfrac{x - x'}{x}\right|}$$

The relative error of the output over the relative error of the input is an important enough concept to have a name: *condition number*

x – true
x' – approx

$$\lim_{x' \to x} \frac{\dfrac{|f(x) - f(x')|}{|f(x)|}}{\left|\dfrac{x - x'}{x}\right|} = \lim_{x' \to x} \left|\frac{x}{x - x'}\right| \cdot \left|\frac{f(x) - f(x')}{f(x)}\right|$$

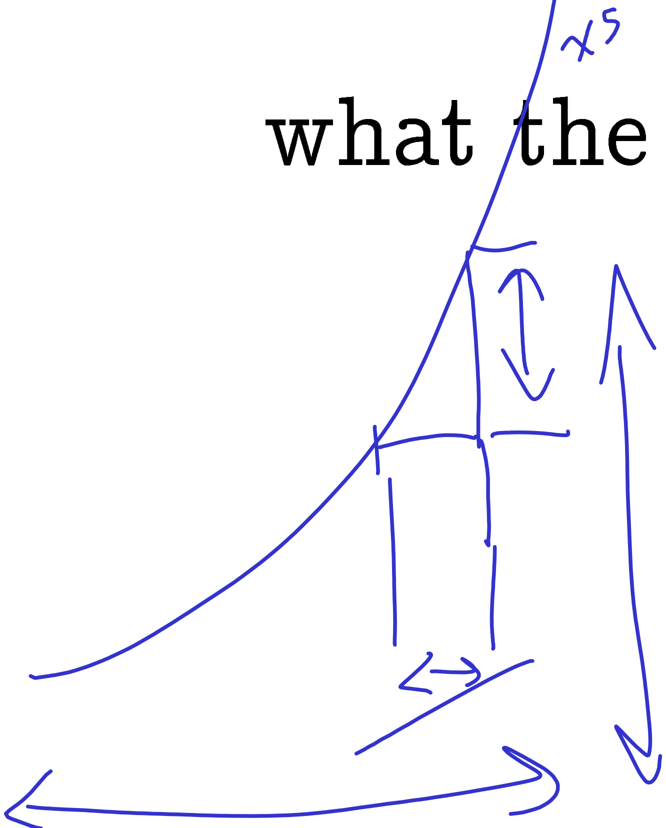
What is the limiting behavior of the condition number as errors get very small?

$$= \lim_{x' \to x} \left|\frac{x}{f(x)}\right| \cdot \left|\frac{f(x) - f(x')}{x - x'}\right| = \left|\frac{x}{f(x)}\right| \lim_{x' \to x} \left|\frac{f(x) - f(x')}{x - x'}\right|$$

$$= \left|\frac{x\, f'(x)}{f(x)}\right|$$

$$f:\ \frac{x(\sqrt{x})'}{\sqrt{x}} = 1/2$$

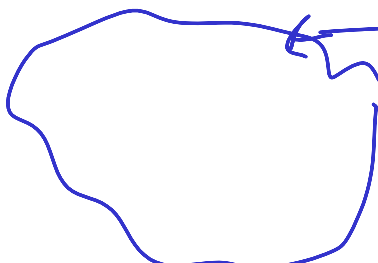
what the condition number means



$$\frac{x f'(x)}{f(x)} = 5$$

$$\left| \frac{x(\cos(x))'}{\cos(x)} \right| = x \tan(x)$$

What's the condition number for $f(x) = x^5$? How about $f(x) = \cos(x)$?
What does this tell you about algorithms to implement f in certain regions?



condition on boundary?

condition > 1 — unstable, any algo unstable

condition < 1 — stable, may be stable alg.

subsequence problem

name $\in \{A, C, T, G\}^n$

How many times does the string AB occur as a subsequence of $ACBCAC$?

str 1 AB

A C B D B C → twice ?

In general, how do you count the number of times string1 occurs as a subsequence of string2?