CSC165 Assignment 2 (due November 13th, 10 pm)

This assignment will give you practice producing proofs and disproofs. A significant part of your work will involve understanding and applying definitions, so you should be sure to start this assignment early. Your proofs must use the proof structure from this course: although symbolic notation may be replaced by precise English prose, you must indicate the steps and scope of your proof using sequences and indentation as we do in class. Each conclusion should be justified using the steps preceding it, given definitions, or elementary mathematical facts (some of which are given in the attached mathematical prerequisites).

Your assignment must be submitted electronically as a PDF file called `a2.pdf`. Although you are welcome to use any word-processing software you choose (no handwriting please) to produce the PDF, you are also welcome to download the LaTeXsource for this assignment and fill in your answers (I'll provide some LaTeXdocumentation and tutorial pointers). You should submit your PDF file early and often to: `https://www.cdf.toronto.edu/students`. You have to use your CDF userid and password for this, so make sure all this is in order well before the due date.

You may work with up to one other person on your assignment, provided you establish your partnership by November 9th. Partnerships are established by each partner submitting a file called `partnerA2` containing the CDF userid of the other partner, to the `https://www.cdf.toronto.edu/students/` web site. Please submit just one copy of the assignment per partnership.

Some questions will use the non-negative rational numbers, denoted $\mathbb{Q}^{\geq 0}$, as the domain. These are ratios of natural numbers where the denominator is non-zero. The positive rational numbers, denoted $\mathbb{Q}^{+}$ are $\mathbb{Q}^{\geq 0} - \{0\}$. In symbols:

$$\begin{aligned} \mathbb{Q}^{\geq 0} &= \{n_1/n_2 \mid n_1, n_2 \in \mathbb{N}, n_2 \neq 0\} \\ \mathbb{Q}^{+} &= \{n_1/n_2 \mid n_1, n_2 \in \mathbb{N}, n_1, n_2 \neq 0\} \end{aligned}$$

1. Prove or disprove the following claims, which are variations on statements about the continuity of $2x$. A good way to begin is to write out the outline of proof structures and disproof structures for each statement, and then try to see which is feasible to complete. If you get stumped writing out the structures, it's time to talk to your instructor or a TA.

   (a) $\exists x \in \mathbb{Q}^{\geq 0}, \exists \epsilon \in \mathbb{Q}^{+}, \forall \delta \in \mathbb{Q}^{+}, \exists y \in \mathbb{Q}^{\geq 0}, |x - y| < \delta \wedge |2x - 2y| \geq \epsilon$.

   (b) $\exists x \in \mathbb{Q}^{\geq 0}, \forall \delta \in \mathbb{Q}^{+}, \exists \epsilon \in \mathbb{Q}^{+}, \exists y \in \mathbb{Q}^{\geq 0}, |x - y| < \delta \wedge |2x - 2y| \geq \epsilon$.

   (c) $\forall \epsilon \in \mathbb{Q}^{+}, \exists \delta \in \mathbb{Q}^{+}, \forall x \in \mathbb{Q}^{+}, \forall y \in \mathbb{Q}^{+}, |x - y| < \delta \Rightarrow |2x - 2y| < \epsilon$.

2. Use the definitions of $\lfloor x \rfloor$, and $\lceil x \rceil$ to prove or disprove the following statements.

   (a) $\forall x \in \mathbb{Q}^{+}, \forall y \in \mathbb{Q}^{+}, \lfloor x \rfloor \cdot \lceil y \rceil \geq \lfloor xy \rfloor$.

   (b) $\exists x \in \mathbb{Q}^{+}, \exists y \in \mathbb{Q}^{+}, \lfloor x \rfloor \cdot \lceil y \rceil > \lceil xy \rceil$.

   (c) $\forall x \in Q^{+}, \forall y \in \mathbb{Q}^{+}, \lfloor x \rfloor \cdot \lfloor y \rfloor \leq \lfloor xy \rfloor$.

3. Use the definitions of $m \bmod 3$ and $n \bmod 5$ from the sheet of mathematical prerequisites to prove or disprove the following statements.

   (a) $\exists n \in \mathbb{N}, n^2 \bmod 5 = 2$.

   (b) $\forall n \in \mathbb{N}, \forall r \in \{0, 1, 2\}, (n^3 \bmod 3 = r) \Rightarrow (n \bmod 3 = r)$.