CSC148 *Intro. to Computer Science*

Lecture 11: *Efficiency of Algorithms, Big O*

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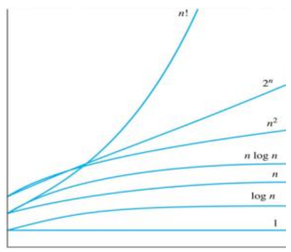
Why efficiency of algorithm matters?

- An example of growth of functions:

n	$\log n$	n	$n \log n$	n^2	2^n	$n!$
10	3×10^{-11} s	10^{-10} s	3×10^{-10} s	10^{-9} s	10^{-8} s	3×10^{-7} s
10^2	7×10^{-11} s	10^{-9} s	7×10^{-9} s	10^{-7} s	4×10^{11} yr	*
10^3	1.0×10^{-10} s	10^{-8} s	1×10^{-7} s	10^{-5} s	*	*
10^4	1.3×10^{-10} s	10^{-7} s	1×10^{-6} s	10^{-3} s	*	*
10^5	1.7×10^{-10} s	10^{-6} s	2×10^{-5} s	0.1 s	*	*
10^6	2×10^{-10} s	10^{-5} s	2×10^{-4} s	0.17 min	*	*

Why efficiency of algorithm matters?

- Another example of growth of functions:



Comparison of growth of functions

- When *n* is arbitrarily big, growth of functions highly depends on the dominant term in the function:
  - $\underline{n}+5$
  - $\underline{n}+1000000$
  - $\underline{n^2}+n+5$
  - $\underline{n^2}+1000000n+5$
  - $2n^2 + \underline{n^3}$
  - $n + \log n + \underline{n \log n}$
  - $n + (\log n)^5 + \underline{n \log n}$
  - $\underline{2^n} + n^2$
  - $\underline{2^n} + n^{200}$

Comparison of growth of functions

- Ignore coefficients as well:
  - $20\underline{n}+5$
  - $200\underline{n}+1000000$
  - $600\underline{n^2}+n+5$
  - $200\underline{n^2}+1000000n+5$
  - $2n^2 + 50\underline{n^3}$
  - $n + 5000 \log n + 300\ \underline{n \log n}$
  - $n + (\log n)^5 + 300\ \underline{n \log n}$
  - $\underline{2^n} + 1000\ n^2$
  - $10002\underline{^n} + 2000\ n^{200}$

Comparison of growth of functions

- Notation:
  - $20\underline{n}+5$ O(n)
  - $200\underline{n}+1000000$ O(n)
  - $600\underline{n^2}+n+5$ O(n^2)
  - $200\underline{n^2}+1000000n+5$ O(n^2)
  - $2n^2 + 50\underline{n^3}$ O(n^3)
  - $n + 5000 \log n + 300\ \underline{n \log n}$ O(n log n)
  - $n + (\log n)^5 + 300\ \underline{n \log n}$ O(n log n)
  - $\underline{2^n} + 1000\ n^2$ O(2^n)
  - $10002\underline{^n} + 2000\ n^{200}$ O(2^n)

Ordering functions by big_O

- Ordering:
  - 20<u>n</u>+5 O(n) 1
  - 200<u>n</u>+1000000 O(n) 1
  - 600<u>n^2</u>+n+5 O(n^2) 3
  - 200<u>n^2</u>+1000000n+5 O(n^2) 3
  - $2n^2$ + 50<u>n^3</u> O(n^3) 4
  - n + 5000 log n + 300 <u>n log n</u> O(n log n) 2
  - n + $(\log n)^5$ + 300 <u>n log n</u> O(n log n) 2
  - <u>2^n</u> + 1000 n^2 O(2^n) 5
  - 1000<u>2^n</u> + 2000 n^{200} O(2^n) 5

Ordering functions by their growth

- Ordering:
- $f_1(n) = (1.5)^n$
- $f_2(n) = 8n^3+17n^2 +111$
- $f_3(n) = (\log n)^2$
- $f_4(n) = 2^n$
- $f_5(n) = \log(\log n)$
- $f_6(n) = n^2 (\log n)^3$
- $f_7(n) = 2^n (n^2 +1)$
- $f_8(n) = n^3+ n(\log n)^2$
- $f_9(n) = 10000$
- $f_{10}(n) = n!$

Time complexity of algorithms

- How time efficient is an algorithm given input size of *n*.
- The ***worst-case time* complexity**:
  - an upper bound on the number of operations an algorithm conducts to solve a problem with input size of *n*.
- We measure time complexity in the order of number of operations an algorithm uses in its worst-case and will demonstrate it using **big_O**.
  - ignore implementation details

Time complexity: Example 1

```
def max(list):
    max = list[0]
    for i in range(len(list)):
        if max < list[i]: max = list[i]
    return max
```

Exact counting:

Count the number of comparisons:

- Assume len(list) = *n*
- The max < list[i] comparison is made *n* times.
- Each time i is incremented, a test is made to see if i < len(list).
- One last comparison determines that i ≥ len(list).
- Exactly **2*n* + 1** comparisons are made.
- Consider the dominant term (as well as ignoring the coefficient)
- Hence, the time complexity of the max algorithm is **O(*n*)**.

Time complexity: Example 2

```
def max2(list):
    max = list[0]
    i=1
    while i< len(list):
        if max < list[i]: max = list[i]
        i+=1
    return max
```

Exact counting:

Count the number of comparisons:

- The max < list[i] comparison is made *n-1* times.
- Each time *i* is incremented, a test is made to see if i < len(list).
- One last comparison determines that i ≥ len(list).
- Exactly 2(n-1) + 1 = **2n -1** comparisons are made.
- Consider the dominant term (as well as ignoring the coefficient)
- Hence, the time complexity of the max2 algorithm is **O(*n*)**.

Time complexity: Example 3

```
def silly(n):
    n = 17 * n**(1/2)
    n = n + 3
    print("n is: {}.".format(n))
    if n > 1997:
        print('very big!')
    elif n > 97:
        print('big!')
    else:
        print('not so big!')
```

Exact counting of the number of comparisons:

- Assume there is not any comparisons inside functions print or format
- Exactly **2** comparisons are made.
- Hence, the time complexity of the silly algorithm is **O(1)**.
- The number of comparisons in print/format is NOT depending on *n*

Estimating big_O

- Instead of calculating the exact number of operations, and then use the dominant term,
- Let's just focus on the dominant parts of the algorithm in the first place.
- Dominant parts of algorithms are **loops** and function **calls.**
- Hence, two things to watch:
  1. We need to **carefully** estimate the number of iterations in the loops in terms of algorithm's input size, i.e. *n*.
  2. If a called function depends on *n* (i.e. it has loops that are in terms of *n*), we should take them into consideration.

Time complexity: Example 1 (revisited)

```
1. def max(list):
2.     max = list[0]
3.     for i in range(len(list)):
4.         if max < list[i]: max = list[i]
5.     return max
```

Calculating big_O:

Focus on the dominant part of the code

(normally loops, also be careful about function calls)

- Assume len(list) = *n*
- The dominant part is the ***for*** loop starting at line 3
  - Line 2 is minor, so is line 1, line 4, and line 5
  - None of these lines have a loop or a function call
- The ***for*** loop in line 3 iterates roughly *n* times
- Hence, the time complexity of the max algorithm is **O(*n*)**.

Time complexity: Example 2 (revisited)

```
1. def max2(list):
2.     max = list[0]
3.     i=1
4.     while i< len(list):
5.         if max < list[i]: max = list[i]
6.         i+=1
7.     return max
```

Calculating big_O:

Focus on the dominant part of the code

- Assume len(list) = *n*
- The dominant part is the ***while*** loop starting at line 4
- This ***while*** loop iterates roughly *n* times
- Hence, the time complexity of the max2 algorithm is **O(*n*)**.

Time complexity: Example 3 (revisited)

```
def silly(n):
    n = 17 * n**(1/2)
    n = n + 3
    print("n is: {}.".format(n))
    if n > 1997:
        print(‘very big!')
    elif n > 97:
        print(‘big!')
    else:
        print('not so big!')
```

Calculating big_O:

Focus on the dominant parts (loops and function calls) of the code

- There is no loop; but there are some function calls
- The number of operations in print/format is NOT depending on *n*
- In other words, these function calls require constant amount of time
- Hence, the overall time complexity of the silly algorithm is **O(1)**.

Time complexity: Example 4

```
def silly2(n):
    n = 17 * n**2
    n = n + 3
    print("n is: {}.".format(n))
    if n > 1997:
       for i in range(n): print(‘so big!‘)
    elif n > 97:
        print(‘big!')
    else:
        print('not so big!')
```

Calculate big_O:

Time complexity: Example 5

```
def silly2(n):
    n = 17 * n**2
    n = n + 3
    print("n is: {}.".format(n))
    if n > 1997:
        print(‘so big!‘)
    elif n > 97:
        for i in range(n): print(‘big!')
    else:
        print('not so big!')
```

Calculate big_O:

Time complexity: Examples 6, 7

- What is the time complexity for this code fragment?

```
sum = 0
for i in range(n//2):
   sum += i * j
```

The loop (roughly) iterates ½ n times:. Hence, it is **O(n)**

- What is the time complexity for this code fragment?

```
sum = 0
for i in range(n//2):
    for j in range(n**2):
       sum += i * j
```

The outer loop iterates $\frac{1}{2}$ n * n^2 times:. Hence, it is **O(n^3)**

Time complexity: Examples 8

- What is the time complexity for this code fragment?

```
sum = 0
for i in range(n//2):
    sum +=i
i = 1
for j in range(n**2):
    sum += i * j
```

The loops iterate $\frac{1}{2}$ n + n^2 times:. Hence, it is **O(n^2)**

Time complexity: Examples 9, 10

- What is the time complexity for this code fragment?

```
sum = 0
if n % 2 == 0:
    for i in range(n*n):
       sum += 1
else:
    for i in range(5, n+3):
       sum += i
```

The loops iterate either n^2 or n+3-5 times. Hence, it is **O(n^2)**

- What is the time complexity for this code fragment?

```
i, sum = 0, 0
while i < n * n:
    sum += i
    i += 1
```

The loop iterate n^2 times:. Hence, it is **O(n^2)**

Time complexity: Examples 11

- What is the time complexity for this code fragment?

```
i, j, sum = 0, 0, 0
while i**2 < n:
    while j**2 < n:
        sum += i * j
        j += 1
    i += 1
```

The outer loop iterates $n^{½}$ * $n^{½}$ times:. Hence, it is **O(n)**

Time complexity: Examples 12

- What is the time complexity for this code fragment?

```
def twoness(n):
   count = 0
   while n > 1:
       n = n // 2
       count = count + 1
return count
```

The loop iterate log n times:. Hence, it is **O(log n)**