CSC236 tutorial exercises, Week #11
best before Thursday evening

These exercises are intended to give you some practice devising deterministic finite state automata (DFAs).

1. Let $L_1 = \{ x \in \{a,b\}^* \mid \text{the number of } a \text{ in } x \text{ is even} \}$, and let $L_2 = \{ z \in \{a,b\}^* \mid |z| \equiv 0 \text{ mod } 3 \}$. Build DFAs that accept $L_1, L_2$, and use the product procedure to build a DFA that accepts $L_1 \cap L_2$.

Sample solution: Here is my specification for $M_1 = \{ Q, \Sigma, \delta, s = q_0, F \}$ that accepts $L_1$:

- $Q = \{E,O\}$,
- $\Sigma = \{a,b\}$,
- $\delta = \begin{bmatrix}
  \delta & E & O \\
  a & O & E \\
  b & E & O
\end{bmatrix}$
- $s = q_0 = E$,
- $F = \{O\}$

Here is my specification for $M_2 = \{ Q, \Sigma, \delta, s = q_0, F \}$ that accepts $L_2$:

- $Q = \{0,1,2\}$,
- $\Sigma = \{a,b\}$,
- $\delta = \begin{bmatrix}
  \delta & 0 & 1 & 2 \\
  a & 1 & 2 & 0 \\
  b & 1 & 2 & 0
\end{bmatrix}$
- $s = q_0 = 0$,
- $F = \{0\}$
Here is my specification for the product machine $M_{1\wedge 2}$ that accepts $L_1 \cap L_2$:

$$
\begin{align*}
Q &= \{(E, 0), (E, 1), (E, 2), (O, 0), (O, 1), (O, 2)\}, \\
\Sigma &= \{a, b\}, \\
\delta &= \begin{array}{cccccc}
\delta(E, 0) & (E, 1) & (E, 2) & (O, 0) & (O, 1) & (O, 2) \\
a & (O, 1) & (O, 2) & (O, 0) & (E, 1) & (E, 2) & (E, 0) \\
b & (E, 1) & (E, 2) & (E, 0) & (O, 1) & (O, 2) & (O, 0)
\end{array}, \\
s &= q_0 = (E, 0), \\
F &= \{ (E, 0) \}.
\end{align*}
$$

2. Use structural induction to prove that the DFAs you propose accept $L_1$ and $L_2$. Without any further induction, prove that your product machine accepts $L_1 \cap L_2$ by constructing a state invariant consisting of conjunctions of the state invariants of the other two machines, and then using your earlier proofs to show that this new state invariant is correct.

**sample solution:** First, define $\Sigma^*$ as the smallest set such that:

(a) $\epsilon \in \Sigma^*$

(b) $s \in \Sigma^* \Rightarrow sa \in \Sigma^* \land sb \in \Sigma^*$

prove that $M_1$ accepts $L_1$: Define $P(s)$ as:

$$
P(s) : \delta^*(q, s) = \begin{cases} 
E & \text{if } s \text{ has an even number of } a \\
O & \text{if } s \text{ has an odd number of } a
\end{cases}
$$

I prove $\forall s \in \Sigma^*, P(s)$ by structural induction.

basis case: $|\epsilon| = 0$, an even number, and $\delta^*(E, \epsilon) = E$ so the implication in the first line of the invariant is true in this case. Also, since $|\epsilon|$ is not odd, the implication in the second line of the invariant is vacuously true. So $P(\epsilon)$ holds.

inductive step: Let $s \in \Sigma^*$ and assume $P(s)$. I will show that $P(sa)$ and $P(sb)$ follow. There are two cases to consider:

**case sa:** Then

$$
\delta^*(E, sa) = \delta(\delta^*(E, s), a) = \begin{cases} 
\delta(E, a) & \text{if } s \text{ has even number of } a \\
\delta(O, a) & \text{if } s \text{ has odd number of } a
\end{cases} \quad \# \text{ by } P(s)
$$

$$
= \begin{cases} 
O & \text{if } sa \text{ has odd number of } a \\
E & \text{if } sa \text{ has even number of } a
\end{cases} \quad \# \text{ one more } a
$$

**case sb:** Then

$$
\delta^*(E, sb) = \delta(\delta^*(E, s), b) = \begin{cases} 
\delta(E, b) & \text{if } s \text{ has even number of } a \\
\delta(O, b) & \text{if } s \text{ has odd number of } a
\end{cases} \quad \# \text{ by } P(s)
$$

$$
= \begin{cases} 
E & \text{if } sb \text{ has even number of } a \\
O & \text{if } sb \text{ has odd number of } a
\end{cases} \quad \# \text{ same number of } a
$$
prove that $M_2$ accepts $L_2$: Define $P(s)$ as:

$$P(s) : \delta^*(q, s) = \begin{cases} 
0 & \text{if } |s| \equiv 0 \mod 3 \\
1 & \text{if } |s| \equiv 1 \mod 3 \\
2 & \text{if } |s| \equiv 2 \mod 3
\end{cases}$$

I prove $\forall s \in \Sigma^*, P(s)$ by structural induction.

basis case: $|s| = 0$, a multiple of 3, and $\delta^*(0, e) = 0$, so the implication in the first line of the invariant is true in this case. Since $|s|$ leaves a remainder of neither 1 nor 2 when divided by 3, the implications on the second and third lines of the invariant are vacuously true. So $P(\epsilon)$ holds.

induction step: Let $s \in \Sigma^*$ and assume $P(s)$. Let $c \in \{a, b\}$. I will prove that $P(sc)$ follows.

$$\delta^*(0, sc) = \delta(\delta^*(0, s), c) = \begin{cases} 
\delta(0, c) & \text{if } |s| \equiv 0 \mod 3 \\
\delta(1, c) & \text{if } |s| \equiv 1 \mod 3 \\
\delta(2, c) & \text{if } |s| \equiv 2 \mod 3
\end{cases}$$

# by $P(s)$

$$= \begin{cases} 
1 & \text{if } |s| \equiv 1 \mod 3 \\
2 & \text{if } |s| \equiv 2 \mod 3 \\
0 & \text{if } |s| \equiv 0 \mod 3
\end{cases}$$

# one more character

So $P(sc)$ follows  

The invariant ensures that all strings with a multiple of 3 characters drive the machine to state 0. The contrapositives of the second and third lines ensure that any string that does not drive the machine to state 1 does not have a length equivalent to 1 mod 3, and any string that does not drive the machine to state 2 does not have a length equivalent to 2 mod 3, so any strings that drive the machine to state 0 have lengths equivalent to 0 mod 3. Hence $M_2$ accepts $L_2$.

prove $M_{1,2}$ accepts $L_1 \cap L_2$: Denote the states for $M_1$ as $Q_1$, the states for $M_2$ as $Q_2$, their respective transition functions as $\delta_1$ and $\delta_2$, and the transition function for $M_{1,2}$ as $\delta_{1,2}$. Inspection of $\delta_{1,2}$ shows that if $(q_1, q_2, c) \in Q_1 \times Q_2 \times \Sigma$, then $\delta_{1,2}((q_1, q_2), c) = (\delta_1(q_1, c), \delta_2(q_2, c))$. Thus the following invariant follows by simply taking conjunctions of the invariants of the component machines, for any $s \in \Sigma^*$

$$P(s) : \delta^*((E, 0), s) = \begin{cases} 
(E, 0) & \text{if } s \text{ has an even number of } a & \land |s| \equiv 0 \mod 3 \\
(E, 1) & \text{if } s \text{ has an even number of } a & \land |s| \equiv 1 \mod 3 \\
(E, 2) & \text{if } s \text{ has an even number of } a & \land |s| \equiv 2 \mod 3 \\
(O, 0) & \text{if } s \text{ has an odd number of } a & \land |s| \equiv 0 \mod 3 \\
(O, 1) & \text{if } s \text{ has an odd number of } a & \land |s| \equiv 1 \mod 3 \\
(O, 2) & \text{if } s \text{ has an odd number of } a & \land |s| \equiv 2 \mod 3
\end{cases}$$

The implication on the first line ensures that all strings with an even number of $a$ and a length that is a multiple of 3 end up in state $(E, 0)$. The contrapositive of the implications on the other
lines ensure that any string that does not drive the machines to one of those 5 states must have an even number of $a$s and a length that is a multiple of 3. Hence $M_{1,2}$ accepts $L_1 \cap L_2$.