UNIVERSITY OF TORONTO

Faculty of Arts and Science

term test #1, Version 2

CSC165H1S

Date: Tuesday October 10, 3:10–4:00pm

Duration: 50 minutes

Instructor(s): Danny Heap

No Aids Allowed

Name:

utorid:

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Please read the following guidelines carefully!

- **Please write your name on both the front and back of this exam.**
- This examination has **4** questions. There are a total of **8 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely. Provide justification unless explicitly asked not to.
- All formulas must have negations applied directly to propositional variables or predicates.
- In your proofs, you may always use definitions of predicates from the course. You may *not* use any external facts about rates of growth, divisibility, primes, or greatest common divisor unless you prove them, or they are given to you in the question.

Take a deep breath.
This is your chance to show us
How much you've learned.

Good luck!

1. **[6 marks] Statements in logic.**

(a) **[3 marks]** Write the truth table for the following formula. No rough work is required.

$$\big((p \Rightarrow q) \vee \neg r\big) \Leftrightarrow \neg p$$

Solution

p	q	r	$\big((p \Rightarrow q) \vee \neg r\big) \Leftrightarrow \neg p$
F	F	F	T
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	F
T	T	T	F

(b) **[3 marks]** Consider the pair of statements:

(1) $\exists n \in \mathbb{N},\ P(n) \Leftrightarrow Q(n)$ (2) $\exists n \in \mathbb{N},\ P(n) \wedge Q(n)$

Define the predicates P and Q with domain $\mathbb{N}$ so that one of these statements is true and the other one false. Note that you're only defining the predicates *once*: the two statements must use the same definitions for P and Q. Also, briefly explain which statement is true and which one is false, and why; no formal proofs necessary.

Solution

Let $P(n)$ be the predicate "$n < 0$" and $Q(n)$ be the predicate "$n < -2$".
The first statement becomes $\exists n \in \mathbb{N}, n < 0 \Leftrightarrow n < -2$, which is true for 0, since false $\Leftrightarrow$ false. The second statement becomes $\exists n \in \mathbb{N}, n < 0 \wedge n < -2$, which is false for every natural number.

2. **[7 marks] Translating statements.**

A **powerful number** is a positive integer m such that for every prime p that divides m, p^2 also divides m.

Express each of the following statements using predicate logic. No justification is required. Note: please review the instructions on the midterm's front page for our expectations in this question. In particular, you may *not* define any helper predicates or sets.

(a) **[2 marks]** 24 is not a powerful number.

Solution

$\exists p \in \mathbb{N}, p \mid 24 \land Prime(p) \land p^2 \nmid 24$

(b) **[5 marks]** 81 is the smallest powerful number greater than 72.

Solution

$\big[\forall p \in \mathbb{N}, \big(p \mid 81 \land Prime(p)\big) \Rightarrow p^2 \mid 81\big] \land \forall n \in \mathbb{N}, \big[n < 81 \land \forall p_0 \in \mathbb{N}, \big(p_0 \mid n \land Prime(p_0)\big) \Rightarrow p_0^2 \mid n\big] \Rightarrow n < 73$

3. **[6 marks] Proofs (inequalities).** Consider the following statement: "For every natural number x there is a natural number y such that $15 > xy > 5$."

   (a) **[1 mark]** Translate the above statement into predicate logic. Use the symbol to denote the set of positive real numbers.

   Solution

   $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, 15 > xy \wedge xy > 5$

   (b) **[1 mark]** Write the negation of this statement, fully-simplified so that all negation symbols are applied directly to predicates. You can simplify $\neg(a > b)$ to $a \leq b$.

   Solution

   $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, 15 \leq xy \vee xy \leq 5$

   (c) **[4 marks]** Disprove the original statement by proving its negation. In your proof, any chains of calculations must follow a top-down order; don't start with the inequality you're trying to prove!

   Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be looked at if your proof is not completely correct.

   Solution

   Proof. Let $x = 15$. Let $y \in \mathbb{N}$. I need to prove that either $xy \geq 15$ or $xy \leq 5$.

   There are two cases to consider.

   Case $y \geq 1$: Then $xy \geq 1 \times x = 15 \geq 15$

   Case $y = 0$: Then $xy = 0 \leq 5$.

   □

4. **[5 marks] Proofs (number theory).** Consider the following statement: "If m and n are integers, and 5 divides both m and n, then 5 divides $2m+n$."

   (a) **[1 mark]** Translate the above statement into predicate logic.

   Solution

   $\forall m,n \in \mathbb{Z}, (5|m \wedge 5 \mid n) \Rightarrow 5 \mid (2m+n)$ also a possible literal translation: $(m,n \in \mathbb{Z} \wedge 5 \mid m \wedge 5 \mid n) \Rightarrow 5 \mid (2m+n)$

   (b) **[4 marks]** Prove the above statement using the definition of divisibility:

   $$x \mid y : \ \exists k \in \mathbb{Z},\ y = kx$$

   Do not use any external facts about divisibility.

   Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be looked at if your proof is not completely correct.

   Solution

   Proof. Let $m,n \in \mathbb{Z}$. Assume $5 \mid m \wedge 5 \mid n$, that is $\exists k_1, k_2 \in \mathbb{Z}, m = 5k_1 \wedge n = 5k_2$. Let k_1, k_2 be such values. Let $k_3 = 2k_1 + k_2$. I need to show that $2m+n = 5k_3$.
   Then,

   $$2m + n \quad = \quad 2(5k_1) + 5k_2 = 5(2k_1 + k_2) = 5k_3. \ \blacksquare$$

   $\square$

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Name:

Question	Grade	Out of
Q1		6
Q2		7
Q3		6
Q4		5
Total		24