University of Toronto
Faculty of Arts and Science

CSC165H1S Midterm 1, Version 1

Date: February 6, 2019 **Duration: 75 minutes** **Instructor(s)**: David Liu, François Pitt

No Aids Allowed

Name: | | | | | | | | | | | | | | | | | | | | | | | | |

Student Number: | | | | | | | | | | | | |

- This examination has **4** questions. There are a total of **8 pages, DOUBLE-SIDED**.
- All statements predicate logic must have negations applied directly to propositional variables or predicates.
- You may not define your own propositional operators, predicates, or sets, unless asked to do so in the question. Please work with the symbols we have introduced in lecture, and any additional definitions provided in the questions.
- Proofs should follow the guidelines used in the course (e.g., explicitly introduce all variables, clearly state all assumptions, justify every deduction in your proof body, etc.)
- In your proofs, you may always use definitions from the course. However, you may **not** use any external facts about these definitions unless the yare given in the question.
- You may **not** use induction for your proofs on this midterm.

Take a deep breath.

This is your chance to show us
How much you've learned.
We **WANT** to give you the credit
That you've earned.
A number does not define you.

Good luck!

Question	Grade	Out of
Q1		8
Q2		7
Q3		6
Q4		5
Total		26

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

1. **[8 marks] Short answers questions.**

(a) **[2 marks]** Let $U = \{a, b, c\}$. Let S_1 be the set of strings over U whose first two letters are the same, and let S_2 be the set of strings over U with length 3. Write down all the elements of $S_1 \cap S_2$.

Solution

$S_1 \cap S_2 = \{aaa, aab, aac, bba, bbb, bbc, cca, ccb, ccc\}$

(b) **[3 marks]** Write down the truth table for the following expression in propositional logic. Rough work (e.g., intermediate columns of the truth table) is **not** required, but can be included if you want.

$$(p \vee q) \Rightarrow \neg r$$

Solution

p	q	r	$(p \vee q) \Rightarrow \neg r$
False	False	False	True
False	False	True	True
False	True	False	True
False	True	True	False
True	False	False	True
True	False	True	False
True	True	False	True
True	True	True	False

(c) **[3 marks]** Consider the following statement (assume predicates P and Q have already been defined):

$$\forall x \in \mathbb{N},\ \exists y \in \mathbb{N},\ P(x, y) \vee Q(x, y)$$

Suppose we want to **disprove** this statement. Write the complete *proof header* for a disproof; you may write statements like "Let $x =$ ____" without filling in the blank. The last statement of your proof header should be "We will prove that..." where you clearly state what's left to prove, in the same style as the lectures or the Course Notes.

You do not need to include any other work (but clearly mark any rough work you happen to use).

Solution

Let $x =$ ____. Let $y \in \mathbb{N}$. We will prove that $P(x, y)$ and $Q(x, y)$ are both False.

2. **[7 marks] Translations.** Let P be the set of all people, and suppose we define the following predicates:

- $Student(x)$: "x is a student", where $x \in P$
- $Attends(x)$: "x attends classes", where $x \in P$
- $Loves(x, y)$: "x loves y", where $x, y \in P$ (note that $Loves(x, y)$ does *not* mean the same thing as $Loves(y, x)$)

Translate each of the following statements into predicate logic. No explanation is necessary. Do not define any of your own predicates or sets. You may use the $=$ and $\neq$ symbols to compare whether two people are the same.

(a) **[1 mark]** There is at least one student who attends class.

Solution

$\exists x \in P,\ Student(x) \land Attends(x)$

(b) **[2 marks]** Every person loves at least one student who attends class.

Solution

$\forall x \in P,\ \exists y \in P,\ Student(y) \land Attends(y) \land Loves(x, y)$

(c) **[2 marks]** Every student who attends class loves himself/herself.

Solution

$\forall x \in P,\ Student(x) \land Attends(x) \Rightarrow Loves(x, x)$

(d) **[2 marks]** For every two distinct (i.e., not equal) people, if the two people love each other, then at most one of them attends class.

Solution

$\forall x, y \in P,\ x \neq y \land Loves(x, y) \land Loves(y, x) \Rightarrow \neg Attends(x) \lor \neg Attends(y)$

3. **[6 marks] A proof about numbers.** Consider the following statement: "For every three integers a, b, and c, if a divides b and b divides c, then a divides c."

(a) **[2 marks]** Translate the above statement into predicate logic. Do *not* use the divisibility predicate |, but instead use the definition of divisibility.

Solution

$\forall a, b, c \in \mathbb{Z},\ \big(\exists k_1 \in \mathbb{Z},\ b = k_1 a\big) \land \big(\exists k_2 \in \mathbb{Z},\ c = k_2 b\big) \Rightarrow \big(\exists k_3 \in \mathbb{Z},\ c = k_3 a\big)$

(b) **[4 marks]** Prove or disprove the above statement. If you choose to disprove the statement, you must start by writing its negation. We have left you space for rough work here and on the next page, but write your formal proof in the box below.

Solution

Proof. Let $a, b, c \in \mathbb{Z}$. Assume that there exists $k_1 \in \mathbb{Z}$ such that $b = k_1 a$, and $k_2 \in \mathbb{Z}$ such that $c = k_2 b$. Let $k_3 = k_1 k_2$. We will prove that $c = k_3 a$.

We begin with our first assumption:

$$
\begin{aligned}
b &= k_1 a \\
k_2 b &= k_2(k_1 a) && \text{(multiplying by } k_2\text{)} \\
c &= k_2(k_1 a) && \text{(by our second assumption)} \\
c &= k_3 a && \text{(by our choice of } k_3\text{)}
\end{aligned}
$$

□

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

4. [**5 marks**] **Floors and Ceilings.** We have the following facts about the floor of a number.

$$\forall x \in \mathbb{R},\ \exists \varepsilon \in \mathbb{R},\ 0 \leq \varepsilon < 1 \land x = \lfloor x \rfloor + \varepsilon \qquad \text{(Fact 1)}$$
$$\forall n \in \mathbb{Z},\ \forall s \in \mathbb{R},\ \ \lfloor n + s \rfloor = n + \lfloor s \rfloor \qquad \text{(Fact 2)}$$

Use these facts to prove the following statement:

$$\forall x, y \in \mathbb{R},\ \ \lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$$

Clearly state where you use each fact in your proof. We have left you space for rough work here and on the next page, but write your formal proof in the box below. HINT: Substitute appropriate expressions for x and y in $\lfloor x + y \rfloor$.

Solution

Proof. Let $x, y \in \mathbb{R}$. By Fact 1, we know that there is some $\varepsilon_1 \in \mathbb{R}$ such that $0 \leq \varepsilon_1 < 1$ and $x = \lfloor x \rfloor + \varepsilon_1$. Similarly, we know that there is some $\varepsilon_2 \in \mathbb{R}$ such that $0 \leq \varepsilon_2 < 1$ and $y = \lfloor y \rfloor + \varepsilon_2$. Then,

$$\lfloor x + y \rfloor = \big\lfloor \lfloor x \rfloor + \varepsilon_1 + \lfloor y \rfloor + \varepsilon_2 \big\rfloor$$

By the definition of *floor* (the greatest integer less than or equal to its input), we know that both $\lfloor x \rfloor$ and $\lfloor y \rfloor$ are integers, and so we can apply Fact 2 to obtain:

$$\begin{aligned} \lfloor x + y \rfloor &= \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \varepsilon_1 + \varepsilon_2 \rfloor \\ &\geq \lfloor x \rfloor + \lfloor y \rfloor && \text{(since } \varepsilon_1, \varepsilon_2 \geq 0) \end{aligned}$$

□

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