UNIVERSITY OF TORONTO

Faculty of Arts and Science

DECEMBER 2017 EXAMINATIONS

CSC165H1F

Duration: 3 hours

Instructor(s): Danny Heap

No Aids Allowed

Name:

Student Number:

Please read the following guidelines carefully!

- This examination has **8** questions. There are a total of **17 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely. Provide justification unless explicitly asked not to.
- All formulas must have negations applied directly to propositional variables or predicates.
- In your proofs, you may always use definitions of predicates from the course. You may *not* use any external facts about rates of growth, divisibility, primes, or greatest common divisor unless you prove them, or they are given to you in the question.
- For algorithm analysis questions, you can jump immediately from a step count to an asymptotic bound without proof (e.g., write "the number of steps is $3n + \log n$, which is $\Theta(n)$").
- You must earn a grade of **at least 40% on this exam to pass this course**.

Take a deep breath.

This is your chance to show us
How much you've learned.

It's been a real pleasure teaching you this term.

Good luck!

Question	Grade	Out of
Q1		3
Q2		9
Q3		7
Q4		11
Q5		10
Q6		6
Q7		6
Q8		12
Total		64

1. **[3 marks] propositions** The truth table below has one column missing:

p	q	r	$(p \Leftrightarrow q) \Leftrightarrow r$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

(a) **[1 mark]** Complete the table by placing either a **T** or a **F** in each row of the empty column.

Solution

p	q	r	$(p \Leftrightarrow q) \Leftrightarrow r$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

part marks: -0.5 for 1-2 errors, -1 for more

(b) **[1 mark]** Write the negation of $(p \Leftrightarrow q) \Leftrightarrow r$ in formal propositional logic.

Solution

$$(p \Leftrightarrow q) \wedge \neg r \vee \neg(p \Leftrightarrow q) \wedge r$$

part marks: they could actually get away with negating the whole thing! -0.5 if it produces 1–2 incorrect results, -1 if more

(c) **[1 mark]** Write an expression equivalent to the negation of $(p \Leftrightarrow q) \Leftrightarrow r$ in formal propositional logic, using only the operations $\wedge, \vee$, and $\neg$, that is you may **not** use $\Rightarrow$ or $\Leftrightarrow$.

Solution

$$(p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$$

part marks: -0.5 if it produces 1–2 wrong results, -1 if more

2. **[9 marks] Primes/Composites** Notice that $2, 5, 11, 17, 23$, and 29 are all primes, and are all congruent to $2 \pmod 3$, that is each of them leaves a remainder of 2 when divided by 3. You may use the following predicates in this question:

$$d \mid n : \exists k \in \mathbb{Z}, n = dk, \text{ where } d, n \in \mathbb{Z}$$
$$Prime(p) : p > 1 \wedge \forall d \in \mathbb{N}, d \mid p \Rightarrow d = 1 \vee d = p, \text{ where } p \in \mathbb{N}$$
$$Composite(n) : \exists d \in \mathbb{N}, d > 1 \wedge d < n \wedge d \mid n, \text{ where } n \in \mathbb{N}$$
$$a \equiv b \pmod m : m \mid (a - b), \text{ where } a, b, m \in \mathbb{Z}, m \neq 0$$

(a) **[1 mark]** Write the following statement in formal predicate logic: "For any natural number n there is a larger natural number p that is both prime and congruent to $2 \pmod 3$."

Solution

$$\forall n \in \mathbb{N}, \exists p \in \mathbb{N}, Prime(p) \wedge p > n \wedge p \equiv 2 \pmod 3$$

A, 1 mark: Correct semantically and formally

(b) **[4 marks]** Prove the statement from the previous part. You may use, without proof, the fact that any natural number greater than 1 may be expressed as the product of 1 or more prime factors.

Solution

Proof: Let $n \in \mathbb{N}$. There are a couple of cases to consider.

Case $n < 3$: Let $p = 5$. Then $Prime(p) \wedge p > n \wedge p \equiv 2 \pmod 3$ (since $3 \mid (5 - 2)$).

Case $n \geq 3$: Let $m = n! - 1$. Since $n \geq 3$ we know $3 \mid n!$, so also $3 \mid (m - 2)$ (result on linear combinations), and $m \equiv 2 \pmod 3$.

Also since $n \geq 3$, $m = n! - 1 \geq 5 > 1$, so m is a product of primes $p_1 \times \cdots \times p_k$. Since if $p_i \equiv 1 \pmod 3$ for all $1 \leq i \leq k$ we would have $m \equiv 1 \pmod 3$, there must be some $p_i \equiv 2 \mod 3$. Let p be such a value. Then $p > n$, since otherwise $p \leq n \Rightarrow p \mid n!$, but $p \mid n! \wedge p \mid (n! - 1) \Rightarrow p \mid 1$ (due to linear combinations), and that's a contradiction. Thus there is a prime p, $p > n \wedge p \equiv 2 \pmod 3$ $\blacksquare$.

A, 1 mark: Introduce names and assumptions

B, 1 mark: Set up an argument (direct, contradiction, induction, whatever...)

C, 2 marks: Successfully derive result

(c) **[1 mark]** Write the following statement in formal predicate logic "For any natural number n there is a larger natural number c that is odd, composite, and congruent to $2 \pmod 3$."

Solution

$$\forall n \in \mathbb{N}, \exists c \in \mathbb{N}, Composite(c) \wedge c > n \wedge c \equiv 1 \pmod 2 \wedge c \equiv 2 \pmod 3$$

A, 1 mark: Correct semantics and formallyf

(d) **[3 marks]** Prove the statement from the previous part.

<u>**Solution**</u>

Proof: Let $n \in \mathbb{N}$. Let $c = 30(n+1)+5$. I will show that c is composite, greater than n, odd, and congruent to $2 \pmod 3$.

By construction $c = 30(n+1)+5 = 5(6n+7)$, so $5 \mid c$ and $5 > 1$ and $5 < c$ (since $6n+7 > 1$). So c is composite.

Also $6n \geq n \Rightarrow 6n+7 > n$, so $c > n$.

Factoring $c = 30(n+1)+5 = 2(15n+17)+1 \equiv 1 \pmod 2$, so c is odd.

Finally $c - 2 = 30(n+1)+3 = 3(10n+11)$, so $3 \mid c-2$ and $c \equiv 2 \pmod 3$. ■

A, 1 mark: Introduce names and assumptions

B, 1 mark: Set up an argument (direct, contradiction, induction, whatever...)

C, 1 mark: Successfully derive result (must show odd, congruence, composite, greater than n)

3. **[7 marks] different moduli** Assume a, b, m, n are integers with $\gcd(m, n) = 1$, $a \equiv b \pmod{m}$, and $a \equiv b \pmod{n}$.

   (a) **[4 marks]** Prove that $a \equiv b \pmod{mn}$. **Hint**: Recall that $a \equiv b \pmod{m}$ means $m \mid (a - b)$. Unwrap the definitions of $m \mid (a - b)$, $n \mid (a - b)$, and $mn \mid (a - b)$. You may use (without proof) the fact that:

$$\forall p, q, r \in \mathbb{Z}, (\gcd(p, q) = 1 \wedge p \mid qr) \Rightarrow p \mid r$$

Solution

Proof: Let $a, b, m, n \in \mathbb{Z}$. Assume that $\gcd(m, n) = 1$, that $a \equiv b \pmod{m}$, and that $a \equiv b \pmod{n}$, that is $\exists k_1, k_2 \in \mathbb{Z}, mk_1 = (a - b) \wedge n_k 2 = (a - b)$. Let k_1, k_2 be such values.
So $k_1 m = k_2 n$, so $m \mid k_2 n$. Since $\gcd(m, n) = 1$, it follows that $m \mid k_2$, that is $\exists k_3 \in \mathbb{Z}, k_3 m = k_2$. Let k_3 be such a value and

$$(a - b) = k_2 n = k_3 mn$$

So $a \equiv b \pmod{mn}$ ■

A, 1 mark: Introduce names and assumptions

B, 1 mark: (try) to use definitions of congruence and division to make progress
-0.5 for examples but not plausible proof structure (0 under C in this case)

C, 2 marks: Success in showing the result. Part marks for making progress or making only arithmetic errors.
-0.5 for getting to $m \mid nk$ but not $m \mid k$.

   (b) **[3 marks]** Prove that if i, j are any integers with $0 \leq i, j < mn$, $i \equiv j \pmod{m}$, and $i \equiv j \pmod{n}$, then $i = j$.

Solution

Proof: Let $i, j, m, n \in \mathbb{Z}$. Assume $\gcd(m, n) = 1$, that $i, j \in [0, mn)$ (half-open interval), that $i \equiv j \pmod{m}$ and $i \equiv j \pmod{n}$. We can assume that $i \geq j$, since the same proof works if we swap them
By the previous part we know that $i \equiv j \pmod{mn}$, so $mn \mid (i - j)$. This means $\exists k \in \mathbb{Z}, mnk = (i - j)$. Let k be such a value. Since i and j lie in the half-open interval $[0, mn)$, and $i \geq j$, their difference is non-negative and at most $(mn - 1) - 0 = mn - 1$, so

$$mnk = (i - j) \Rightarrow 0 \leq mnk \wedge mnk < mn \Rightarrow k = 0$$

Thus $i = j$ ■

A, 1 mark: Introduce names and assumptions. They may say the assumptions on m, n are the same as the last part.

B, 1 mark: (try) to use equivalence $\pmod{mn}$, or perhaps some other argument that they structure...

C, 1 mark: Successfully use equivalence $\pmod{mn}$, or some other argument, to show $i = j$.

4. **[11 marks] step counting** Consider the gcd function: Consider the Fibonacci sequence, f_n defined by:

$$f_n = \begin{cases} n & \text{if } n < 2 \\ f_{n-2} + f_{n-1} & \text{if } n \geq 2 \end{cases}$$

(a) **[3 marks]** Use induction on n to prove that for every natural number n greater than 2, $f_n > f_{n-1} \wedge f_{n-1} > 0$.

Solution

Proof (induction): Define $P(n) : n > 2 \Rightarrow f_n > f_{n-1} \wedge f_{n-1} > 0$. I prove that $\forall n \in \mathbb{N}, P(n)$.

base case: From the definition $f_3 = 2 > 1 = f_2$, and $f_2 = 2 > 0$, so $P(3)$ is true.

inductive step: Let $n \in \mathbb{N}$. Assume $P(n)$. I want to show that $P(n+1)$ follows.

Assume that $n \geq 3$, since otherwise (given the base case) there is nothing to prove. That means that

$$f_{n+1} = f_{n-1} + f_n > f_n \qquad \text{(by IH, } f_{n-1} > 0\text{)}$$
$$\wedge 0 < f_{n-1} < f_n \qquad \text{(by IH, } f_n > f_{n-1}\text{)}$$

So $P(n+1)$ follows ■

A, 1 mark: introduce names and assumptions, base case

B, 1 mark: inductive step, including IH

C, 1 mark: derive result

(b) **[4 marks]** Use induction on n to prove that for every natural number n, $\gcd(f_n, f_{n+1}) = 1$.

Solution

Proof: Define $P(n) : \gcd(f_n, f_{n+1}) = 1$. I will prove that $\forall n \in \mathbb{N}, P(n)$.

base case: $\gcd(f_0, f_1) = \gcd(0, 1) = 1$, so $P(0)$ is true.

inductive step: Let $n \in \mathbb{N}$ and assume $P(n)$. I want to show that $P(n+1)$ follows.

Since $f_{n+2} = f_n + f_{n+1}$, any integer that divides both f_{n+2} and f_{n+1} also divides their difference, $f_{n+2} - f_{n+1} = f_n$ (divisibility of linear combinations). The largest integer that divides both f_n and f_{n+1} is their greatest common divisor 1, so $\gcd(f_{n+1}, f_{n+2}) = 1$ ■

A, 1 mark: introduce names and assumptions

B, 1 mark: base case

C, 1 mark: inductive step, including IH

D, 1 mark: derive result

(c) **[4 marks]** Read over the function gcd below:

```python
def gcd(n, m):
    while m * n != 0:
        n, m = m, n % m
    if m == 0:
        return n
    else:
        return m
```

Assume the body of the loop in gcd is one step. Use induction on n to prove that for any natural number n greater than 2, $\text{gcd}(f_n, f_{n-1})$ takes at least $n-2$ steps, where f_n is the nth number in the Fibonacci sequence defined above. If m and n are natural numbers with $m \neq 0$, you may assume that there is some integer q such that $n = qm + (n\%m)$, that $m > (n\%m)$ and $(n\%m) \geq 0$.

Solution

Proof (induction): Define

$$P(n) : n > 2 \Rightarrow \text{gcd}(f_n, f_{n-1}) \text{ takes at least } n-2 \text{ steps}$$

base case: $\text{gcd}(2, 1) = \text{gcd}(f_3, f_2)$. After the first iteration of the loop, n is set to 1 and m is set to $2\%1 = 0$, so there is $1 = 3 - 2$ iteration of the loop, so $P(3)$ is true.

inductive step: Let $n \in \mathbb{N}$ and assume $P(n)$. I will show that $P(n+1)$ follows from this, that is $\text{gcd}(f_{n+1}, f_n)$ takes at least $n + 1 - 2 = n - 1$ steps.
After the first iteration of the loop n is set to f_n and m is set to $f_{n+1}\%f_n$. Notice that f_{n-1} satisfies the conclusion of the Quotient-Remainder Theorem:

$$f_{n+1} = 1 \times f_n + f_{n-1} \wedge f_n > f_{n-1} \wedge f_{n-1} \geq 0$$

So at the end of the first loop $(n, m) = (f_n, f_{n-1})$. By the inductive hypothesis, there will be at least $n-2$ to complete the work of $\text{gcd}(f_n, f_{n-1})$, so there will be $n-1$ steps in all
■

A, 1 mark: introduce names and assumptions

B, 1 mark: base case

C, 1 mark: set up inductive step and IH

D, 1 mark: successfully derive result. -1 if they don't tie induction to loop iterations somehow. -1 if they don't tie values to fibonacci numbers

5. **[10 marks] runtime** Function `f1` takes positive integer n as input, and its runtime depends only on n:

(a) **[4 marks]**

```
1 def f1(n):
2   i = 0
3   while i**2 < n:
4       j = 0
5       while j < n:
6           j = j + 3
7       i = i + 2
```

We will consider the runtime of `f1` to be the total cost of executing Line 6 over all loop iterations, and ignoring all other operations. Determine the exact number of times Line 6 is executed in terms of the input size, n.

Solution

sample: The inner loop sets j to $3s$ for every step s until $j \geq n$, so it executes $\lceil n/3 \rceil$ for each i. The outer loop sets i to $2s'$ for every step s', until $i^2 \geq n$. Thus it exits when $4s'^2 \geq n$, or $s' = \lceil \sqrt{n}/2 \rceil$.
Combining these gives:

$$\left\lceil \frac{n}{3} \right\rceil \left\lceil \frac{\sqrt{n}}{2} \right\rceil$$

... steps for input size n.

A, 1 mark: expression appears to be, at least, a product

B, 2 marks: expression takes into account steps of 2 and 3, somehow

C, 1 mark: expression takes into account outer square root

(b) **[1 mark]** Use your answer from (a) to determine a simple Theta expression for the runtime of `f1`. No justification required.

Solution

sample: Floors and ceilings, as well as multiplicative coefficients, do not change Θ, so we have

$$RT \in \Theta(n\sqrt{n}) = \Theta(n^{3/2})$$

part marks: base this on their expression for part a. -0.5 if they don't remove floor/ceiling. -0.5 if they don't remove multiplicative coefficients. -0.5 (but never go below zero...) if they end up in a different complexity class.

(c) **[4 marks]** Helper function `h(k)` takes k steps for input of size k. For example, `h(10)` takes 10 steps. Consider the runtime of `f2` to be the total cost of executing Line 4 over all loop iterations.

```
1 def f2(n):
2   i = 0
3   while i**2 < n:
4       h(i)
5       i = i + 1
```

Recall the formula, valid for all $j \in \mathbb{N}$:

$$\sum_{i=0}^{j} i = \frac{j(j+1)}{2}$$

Determine the exact cost of executing line 4 of function `f2` above.

Solution

sample: Helper function h executes $h(i)$ steps as i ranges from 0 to $\lceil\sqrt{n}\rceil - 1$, contributing

$$\sum_{j=0}^{\lceil\sqrt{n}\rceil-1} j = \frac{(\lceil\sqrt{n}\rceil - 1)(\lceil\sqrt{n}\rceil)}{2} \text{ steps}$$

A, 2 marks: expression takes into account $i^2 < n$

B, 1 mark: expression takes into account $h(i)$

C, 1 mark: expression is an integer

(d) **[1 mark]** Use (c) to determine a simple Theta expression for the runtime of `f2`. No justification required.

Solution

sample: Floors, ceilings, slower-growing terms, and multiplicative constants do not change Θ, so we have

$$RT \in \Theta(n)$$

part marks: base this on previous part. -0.5 if they don't remove floor/ceiling. -0.5 if they don't remove constant. -0.5 (but never below zero...) if they end up in a different complexity class.

6. **[6 marks] average case analysis**

Assume that p is a program, that $\mathcal{I}_{p,n}$ is the set of inputs of size n for p, that for all n, $|\mathcal{I}_{p,n}| \in \mathbb{Z}^+$ (in other words, for each n there are finitely many inputs for p), and that $RT(x)$ is the number of steps $p(x)$ requires to run.

Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and assume:

$$\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \forall x \in \mathcal{I}_{p,n},\ n \geq n_1 \Rightarrow RT(x) \leq c_1 f(n)$$
$$\exists c_2, n_2 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \forall x \in \mathcal{I}_{p,n},\ n \geq n_2 \Rightarrow RT(x) \geq c_2 f(n)$$

Define the average runtime for inputs of size n as:

$$AVG(n) = \frac{\sum_{x \in \mathcal{I}_{p,n}} RT(x)}{|\mathcal{I}_{p,n}|}$$

Prove that $AVG \in \Theta(f)$, in other words prove:

$$\exists c_3, c_4, n_3 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_3 \Rightarrow c_3 f(n) \leq AVG(n) \wedge AVG(n) \leq c_4 f(n)$$

Solution

Proof: Let $\mathcal{I}_{p,n}$ and RT be as defined above, and assumptions about f, c_1, n_1, c_2, n_2 be as above. Let $n' = \max(n_1, n_2)$, let $c_3 = c_1$ and let $c_4 = c_2$.

Then $\forall n \in \mathbb{N}, n \geq n'$ implies:

$$AVG(n) = \frac{\sum_{x \in \mathcal{I}_{p,n}} RT(x)}{|\mathcal{I}_{p,n}|} \leq \frac{\sum_{x \in \mathcal{I}_{p,n}} c_1 f(n)}{|\mathcal{I}_{p,n}|} = \frac{|\mathcal{I}_{p,n}| c_1 f(n)}{|\mathcal{I}_{p,n}|} = c_1 f(n) = c_3 f(n)$$

(by assumption)

$$AVG(n) = \frac{\sum_{x \in \mathcal{I}_{p,n}} RT(x)}{|\mathcal{I}_{p,n}|} \geq \frac{\sum_{x \in \mathcal{I}_{p,n}} c_2 f(n)}{|\mathcal{I}_{p,n}|} = \frac{|\mathcal{I}_{p,n}| c_2 f(n)}{|\mathcal{I}_{p,n}|} = c_2 f(n) = c_3 f(n)$$

(by assumption)

$$\Rightarrow AVG \in \Theta(f) \quad \blacksquare$$

A, 2 mark: Introduce names and assumptions. They are allowed to “borrow” these introductions from the question statement explicitly. There are a lot of names, due to the nature of the question

B, 2 marks: Set up an argument (direct, contradiction, induction, whatever...).

C, 2 marks: Successfully arrive at the conclusion.

7. **[6 marks] connectedness**

Recall the definition of the degree, $d(u)$ of vertex u in a graph $G = (V, E)$:

$$d(u) = |\{(u, v) \mid (u, v) \in E\}|$$

Recall also that there is a path between vertices u and v if there is a sequence of **distinct** vertices $v_0, v_1, \ldots, v_k$, where $v_0 = u$ and $v_k = v$, and for every $i \in \{0, \ldots, k-1\}$ there is an edge $(v_i, v_{i+1}) \in E$.

Finally, recall that G is **connected** means that for any pair of vertices $u, v \in V$, there is a path from u to v.

Prove that for every graph $G = (V, E)$, if $\forall v \in V, d(v) \geq \lfloor |V|/2 \rfloor$, then G is connected.

Solution

Proof: Let $G = (V, E)$ be an arbitrary graph and assume $\forall v \in V, d(v) \geq \lfloor |V|/2 \rfloor$. Let $u, v \in V$ be an arbitrary pair of vertices in G. I will show that there is a path from u to v.

Let $N(u) = \{v \mid v \in V \wedge (u, v) \in E\} \cup \{u\}$ and $N(v) = \{w \mid w \in V \wedge (v, w) \in E\} \cup \{v\}$. By assumption $|N(u)| \geq \lfloor |V|/2 \rfloor + 1$ and $|N(V)| \geq \lfloor |V|/2 \rfloor + 1$. I will show that $N(u) \cap N(v) \neq \emptyset$, so there is at least one vertex in common, forming a path from u to v. There are two cases to consider, depending on whether $|V|$ is even or odd:

Case $|V|$ is even: Counting vertices, we have:

$$|N(v)| + |N(u)| \geq \lfloor |V|/2 \rfloor + \lfloor |V|/2 \rfloor + 2 = |V| + 2$$

So $N(u)$ and $N(v)$ must have at least 2 vertices in common, since $N(u) \cup N(v) \subseteq V$. ■

Case $|V|$ is odd: Let $k_1 \in \mathbb{N} = (|V| - 1)/2$, so $|V| = 2k + 1$. Then, counting vertices, we have:

$$|N(v)| + |N(u)| \geq \lfloor |V|/2 \rfloor + \lfloor |V|/2 \rfloor + 2 = 2k + 2 = |V| + 1$$

So $N(u)$ and $N(v)$ must have at least 1 vertex in common, since $N(u) \cup N(v) \subseteq V$. ■

A, 1 mark: Introduce names and assumptions. They will probably need to introduce V and E, and perhaps some arbitrary vertices. Do **not** deduct if they treat a name as being introduced by an assumption, e.g. $\exists k, m = 3k$, and then go on to use k as though it were introduced.

B, 2 marks: Set up an argument. This may not be the same as our approach, but give them up to 2 for setting up proof by induction, contradiction, properly, even if they cannot successfully show connectivity. Give part credit for knowing what they need to show.

C, 3 marks: Show connectivity. Give no more than 1.5/3 if they don't convince you the graph is connected.

8. **[12 marks] cycles**

Recall the definitions of **degree**, **path**, and **connected** from the previous question, and recall that a sequence of vertices $v_0, \ldots, v_k$ in graph $G = (V, E)$ is a cycle if:

- the sequence is a path from v_0 to v_k containing at least 3 distinct vertices, and
- there is an edge $(v_k, v_0) \in E$.

(a) **[3 marks]** Prove that for every graph $G = (V, E)$, if $\exists v \in V, d(v) \geq 3$, then $|V| \geq 4$.

Solution

Proof: Let $G = (V, E)$. Assume $\exists v \in V, d(v) \geq 3$. Let v be such a vertex. I will show that $|V| \geq 4$.
Let $N(v) = \{u \mid (u, v) \in E\} \cup \{v\}$. Since $d(v) \geq 3$ $|N(u)| \geq 4$ and $N(u) \subseteq V$ ■

A, 1 mark: Introduce names and assumptions

B, 1 mark: Set up an argument (direct, induction, contradiction, whatever...)

C, 1 mark: Successfully derive result.

(b) **[4 marks]** Prove that for every graph $G = (V, E)$, if $\forall v \in V, d(v) \geq 3$, then for every $v \in V$ there is a path consisting of at least 4 distinct vertices starting at v.

Solution

Proof: Let $G = (V, E)$ and assume that $\forall v \in V, d(v) \geq 3$. Let $v \in V$. I will show that there is a path beginning at v with at least 4 vertices.
Let $P(v)$ be a maximal-length path beginning at v, $P(v) = v, v_1, \ldots, u$, let u be the final vertex in $P(v)$. Since, by assumption, $d(u) \geq 3$ it must have at least 3 edges leading to other vertices in $P(v)$, since otherwise it would have an edge creating a longer path. This means that $P(v)$ has at least 4 vertices, including u ■

A, 1 mark: Introduce names

B, 1 mark: Set up a proof structure

C, 2 marks: Derive conclusion

(c) **[5 marks]** Prove that for every graph $G = (V, E)$, if $\forall v \in V, d(v) \geq 3$, then G contains at least one cycle containing at least 4 distinct vertices.

<u>**Solution**</u>

Proof: Let $G = (V, E)$ and assume that $\forall v \in V, d(v) \geq 3$. Let $v \in V$. I will show that there is a cycle with at least distinct vertices starting at v.

From the previous part we know that there is a path $P(v)$ with at least 4 vertices. Let u be the farthest vertex in $P(v)$ from v. Since, by assumption, $d(u) \geq 3$ and it has no edge to a vertex outside $P(v)$, u must have at least 3 neighbours in $P(v)$. Let u' be v's predecessor in $P(v)$ and u'' be u''s predecessor. Let u''' be a neighbour of u that is different fromf u' and u''. Then $u, u''', \ldots, u'', u', u$ forms a cycle of length at least 4 ■

A, 1 mark: Introduce names

B, 1 mark: Set up a proof structure

C, 3 marks: Derive conclusion

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

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