Learning Objectives

By the end of this worksheet, you will:

- Analyse the running time of loops whose loop counter changes differently in different iterations.
- Analyse the running time of functions that call helper functions.
- Understand and express properties about the minimum and maximum of a set of numbers.

1. Varying loop increments. In lecture, we saw one (complicated) example of a loop where the change to the loop variable was not the same on each iteration. In this question, you’ll get some practice analyzing such loops yourself using a general technique. For each of the following functions, do the following:

   (i) Identify the minimum and maximum possible change for the loop variable in a single iteration.
   (ii) Use this to determine formula for an exact lower bound and upper bound on the value of the loop variable after \( k \) iterations. (E.g., “\( i_k \geq k \)” and “\( i_k \leq 2^k \)”.)
   (iii) Use these formulas and the loop condition to bound the exact number of loop iterations that will occur.\(^1\)
   - Use the lower bound on the loop variable (e.g., \( i_k \geq k \)) to find a value for \( k \) that guarantees that the loop condition is false. This gives you an upper bound on the possible number of iterations.
   - Use the upper bound on the loop variable (e.g., \( i_k \leq 2^k \)) to find a value of \( k \) that guarantees that the loop condition is true. This gives you a lower bound on the possible number of iterations.
   (iv) Use your upper and lower bounds on the number of iterations to find Big-Oh and Omega bounds on the running time of the function. Note that if you have the same expression for Big-Oh and Omega, then you can also conclude a Theta bound.

```python
def varying1(n: int) -> None:
    i = 0
    while i < n:
        if i % 3 == 0:
            i = i + 1
        elif i % 2 == 1:
            i = i + 3
        else:
            i = i + 6
```

Solution

The minimum change in the loop is that \( i \) increases by 1; the maximum change is that \( i \) increases by 6. Let \( i_k \) be the value of \( i \) after \( k \) iterations. The previous observation tells us that \( k \leq i_k \leq 6k \) (these are the lower and upper bounds on \( i_k \)).

Part 1 (upper bound on runtime).

We want to determine a good upper bound (“at most ____”) on the number of iterations that could occur before the loop stops. Since the loop terminates when \( i \geq n \), we want to find the smallest value of \( k \) such that \( i_k \geq n \).

To do this, we use the lower bound on \( i_k \): since we know that \( i_k \geq k \), substituting \( k = n \) gives us the desired inequality \( i_k \geq n \). This means that the loop can run at most \( n \) iterations (for \( k = 0, 1, \ldots, n-1 \)).\(^2\)

Since each iteration takes 1 step, this loop takes at most \( n \) steps, leading to an upper bound on the running time of \( \mathcal{O}(n) \).

Part 2 (lower bound on runtime).

\(^1\)This works whenever the loop condition is of the form \( i < \_ \), where \( i \) is the loop variable.

\(^2\)This is the true lower bound—some iterations may finish before the loop condition is satisfied.
Now we want to determine a good lower bound ("at least _____") on the number of iterations that must occur before the loop stops. Since the loop terminates when \( i \geq n \), we want to find the largest value of \( k \) such that \( i_k < n \) (i.e., that forces the loop to keep going).

Since \( i_k \leq 6k \), we want to find the largest value of \( k \) where \( 6k < n \), or equivalently \( k < \frac{n}{6} \). Note the strict inequality here: this means that we must use \( k = \lceil \frac{n}{6} \rceil - 1 \) to get the largest integer less than but not equal to \( \frac{n}{6} \).

So what we’ve shown is that after \( \lceil \frac{n}{6} \rceil - 1 \) loop iterations, \( i \) is still less than \( n \), and so at least this many loop iterations (and therefore steps) must occur. Since \( \lceil \frac{n}{6} \rceil - 1 \in \Omega(n) \), we can conclude that a lower bound on the running time of \( \Omega(n) \).

Note: since the upper bound and lower bounds are the same function in the Big-Oh/Omega, we can conclude that the overall running time of this function is \( \Theta(n) \).

\*Note that \( k = n \) is not a loop iteration, since this is the point where the loop condition becomes false.

\*Using \( k = \lfloor \frac{n}{6} \rfloor \) would make this an equality when \( n \) is a multiple of 6.

### Solution

The argument is the same as the previous one, except now \( i \) increases by at least a multiplicative factor of 2, and at most a factor of 3. This means that \( 2^k \leq i_k \leq 3^k \), and so the number of iterations is at most \( \lceil \log_2 n \rceil \) and at least \( \lceil \log_3 n \rceil - 1 \) [using the same reasoning as in part (a)].

That is, the upper bound on the running time here is \( O(\log_2 n) \), and the lower bound is \( \Omega(\log_3 n) \). Since we know that \( \log_3 n \in \Theta(\log_2 n) \), we can conclude that the tight bound on the running time is \( \Theta(\log n) \).

2. Helper functions. We have mainly analysed loops as the mechanism for writing functions whose running time depends on the size of the function’s input. Another source of non-constant running times that you often encounter are other functions that are used as helpers in an algorithm.

For this exercise, consider having two functions helper1 and helper2, which each take in a positive integer as input. Moreover, assume that helper1’s running time is \( \Theta(n) \) and helper2 is \( \Theta(n^2) \), where \( n \) is the value of the input to these two functions.

Your goal is to analyse the running time of each of the following functions, which make use of one or both of these helper functions. When you count costs for these function calls, simply substitute the value of the argument of the call into the function \( f(x) = x \) or \( f(x) = x^2 \) (depending on the helper). For example, count the cost of calling helper1\( (k) \) as \( k \) steps, and helper2\( (2*n) \) as \( 4n^2 \) steps.

```python
def varying2(n: int) -> None:
    i = 1
    while i < n:
        if n % i <= i/2:
            i = 2 * i
        else:
            i = 3 * i
```

Solution

```python
def f1(n: int) -> None:
    helper1(n)
    helper2(n)
```

Solution

The call to helper1 takes \( n \) steps, and the call to helper2 takes \( n^2 \) steps, for a total of \( n^2 + n \) steps.
This is $\Theta(n^2)$.

```python
def f2(n: int) -> None:
    i = 0
    while i < n:
        helper1(n)
        i = i + 2
    j = 0
    while j < 10:
        helper2(n)
        j = j + 1
```

**Solution**

The first loop takes $\left\lfloor \frac{n}{2} \right\rfloor$ iterations, and each iteration requires $n$ steps for the call to `helper1`. As with nested loops, we ignore the lower-order cost of the loop counter increment $i = i + 2$. So the total cost of this loop is $\frac{n}{2} \cdot n$.

The second loop runs for 10 iterations, and each iteration requires $n^2$ steps for the call to `helper2`. So we count the cost for this loop as $10n^2$.

The total cost is $\left\lfloor \frac{n}{2} \right\rfloor \cdot n + 10n^2$, which is $\Theta(n^2)$.

```python
def f3(n: int) -> None:
    i = 0
    while i < n:
        helper1(i)
        i = i + 1
    j = 0
    while j < 10:
        helper2(j)
        j = j + 1
```

**Solution**

The first loop takes $n$ iterations, but now the cost of the call to `helper1` changes at each iteration. For a fixed iteration of this loop, the cost of calling `helper1(i)` is $i$, and so the total cost over all iterations of this loop is $\sum_{i=0}^{n-1} i = \frac{(n-1)n}{2}$ (note that this is the same as when we analysed one of the nested loop examples from lecture).

Similarly, the cost of the second loop is $\sum_{j=0}^{9} j^2$; this is one, however, is a constant cost with respect to $n$.

So the first loop has a running time of $\Theta(n^2)$, and the second has a running time of $\Theta(1)$. The overall running time is the sum of these two, which is $\Theta(n^2)$.
3. **A more careful analysis.** Recall this function from lecture:

```python
def f(n: int) -> None:
    x = n
    while x > 1:
        if x % 2 == 0:
            x = x // 2
        else:
            x = 2*x - 2
```

We argued that for any positive integer value for $x$, if two loop iterations occur then $x$ decreases by at least one. This led to an upper bound on the running time of $O(n)$, but it turns out that we can do better.

(a) First, prove that for any positive integer value of $x$, if three loop iterations occur then $x$ decreases by at least a factor of 2. Note: this is an exercise in covering all possible cases; it's up to you to determine exactly what those cases are in your proof.

**Solution**

**Proof.** Let $x_0$ be the starting value of $x$, and $x_1$, $x_2$, and $x_3$ be the value of $x$ after 1, 2, and 3 loop iterations, respectively. We want to prove that $x_3 \leq \frac{1}{2}x_0$. There are many ways of dividing this proof into cases based on whether these values are even/odd. One simple approach is to look at the remainders of $x_0$ when dividing by 8; this has a lot of cases, but the calculation required for each case is pretty straightforward. Here’s one as an example.

**Case ??** assume $x$ has remainder 5 when divided by 8, i.e., that there exists $k \in \mathbb{Z}$ such that $x_0 = 8k + 5$. In this case, $x_0$ is odd, and so Line 7 executes in the first loop iteration, making

$$x_1 = 2x_0 - 2 = 16k + 8 = 2(8k + 4)$$

So then $x_1$ is even, and at the second loop iteration Line 5 executes, making

$$x_2 = \frac{1}{2}x_1 = 8k + 4 = 2(4k + 2)$$

So then $x_2$ is even, and at the third loop iteration Line 5 executes again, making

$$x_3 = \frac{1}{2}x_2 = 4k + 2 = \frac{1}{2}x_0 - \frac{1}{2}$$

Therefore $x_3 \leq \frac{1}{2}x_0$. □

(b) For every $k \in \mathbb{N}$, let $x_k$ be the value of the variable $x$ after $3k$ loop iterations, in the case when $3k$ iterations occur. Using part (a), find an upper bound on $x_k$, and hence on the total number of loop iterations that will occur (in terms of $n$). Finally, use this to determine a better asymptotic upper bound on the runtime of $f$ than $O(n)$.

(Note: you might need to write your analysis on a separate sheet of paper.)

**Solution**

We showed in part (a) that after 3 iterations, the current value of $x$ decreases by at least a factor of 2, or the loop has terminated. So then for any $k$, either the loop terminates within $3k$ iterations, or the value of $x$ has decreased by at least a factor of $2^k$. Since $x$ is initialized to $n$, we know that $x_k \leq \frac{n}{2^k}$.

The loop terminates when $x \leq 1$, and this occurs when $2^k \geq n$, i.e., $k \geq \lceil \log n \rceil$. So then the loop will run

\footnote{We phrase this as a conditional because it might be the case that the loop stops after fewer than two iterations.}
for at most $3 \cdot \log n$ iterations; since each iteration takes constant time, the total runtime is $O(\log n)$. 