Learning Objectives

By the end of this worksheet, you will:

- Prove statements using the definition of Big-Oh and its negation.
- Represent constant functions in Big-Oh expressions.
- Understand and use the definition of Omega and Theta to compare functions.

For your reference, here is the formal definition of Big-Oh:

\[ g \in \mathcal{O}(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n) \]

1. Constant functions

As we discussed in class, constant functions, like \( f(n) = 100 \), will play an important role in our analysis of running time next week. For now let’s get comfortable with the notation.

(a) Let \( g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} \). Show how to express the statement \( g \in \mathcal{O}(1) \) by expanding the definition of Big-Oh. 

**Solution**

\[ g \in \mathcal{O}(1) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq c. \]

(b) Prove that \( 100 + \frac{77}{n+1} \in \mathcal{O}(1) \).

Note: this proof isn’t too mathematically complex; treat this as another exercise in making sure you understand the definition of Big-Oh!

Hint: one algebraic property you can use is that \( \forall x, y \in \mathbb{R}^+, x \geq y \Rightarrow \frac{1}{x} \leq \frac{1}{y} \).

**Solution**

We want to prove that \( \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow 100 + \frac{77}{n+1} \leq c. \)

There are many possible choices of \( c \) and \( n_0 \) here. One possibility is \( c = 101 \) and \( n = 76 \). We leave the calculation as an exercise.

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1 Remember that we often abbreviate Big-Oh expressions to just show the function bodies. “\( \mathcal{O}(1) \)” is really shorthand for “\( \mathcal{O}(f) \), where \( f \) is the constant function \( f(n) = 1 \).”
2. **Omega.** Recall that we can think of Big-Oh notation as describing an upper bound on the rate of growth of a function: saying \( g \in \mathcal{O}(f) \) is like saying \( g \) grows at most as fast as \( f \). As we saw in class, sometimes we care just as much about a lower bound on the rate of growth and for this, we have the symbol \( \Omega \) (the Greek letter Omega), which is defined analogously to Big-Oh:

\[
g \in \Omega(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \geq cf(n)
\]

Using this definition, prove that for all \( f, g : \mathbb{N} \to \mathbb{R}_{\geq 0} \), if \( g \in \mathcal{O}(f) \), then \( f \in \Omega(g) \).

**Solution**

**Proof.** Let \( f, g : \mathbb{N} \to \mathbb{R}_{\geq 0} \). Assume that \( g \in \mathcal{O}(f) \), i.e., that there exist \( c_1, n_1 \in \mathbb{R}^+ \) such that for all \( n \in \mathbb{N} \), if \( n \geq n_1 \) then \( g(n) \leq c_1 f(n) \). We want to prove that there exist \( c_2, n_2 \in \mathbb{R}^+ \) such that for all \( n \in \mathbb{N} \), if \( n \geq n_2 \) then \( f(n) \geq c_2 g(n) \).

Let \( c_2 = \frac{1}{c_1} \) and \( n_2 = n_1 \). Let \( n \in \mathbb{N} \), and assume that \( n \geq n_2 \). We want to prove that \( f(n) \geq c_2 g(n) \).

Since \( n_2 = n_1 \), we know from our assumption that \( n \geq n_1 \). So then by our first assumption (that \( g \in \mathcal{O}(f) \)), we know that \( g(n) \leq c_1 f(n) \). Dividing both sides by \( c_1 \) yields \( \frac{1}{c_1} g(n) \leq f(n) \), and so \( c_2 g(n) \leq f(n) \). \( \Box \)
3. Theta. As we saw in class, both Big-Oh and Omega are limited in the same way as inequalities on numbers. “$2 \leq 10^{10}$” is a true statement, but not very insightful; similarly, “$n + 1 \in O(n^{10})$” and “$2^n + n^2 \in \Omega(n)$” are both true, but not very precise.

Our final piece of notation is the symbol $\Theta$ (the Greek letter Theta), which we defined in rather simple terms:

$$g \in \Theta(f) : g \in O(f) \land g \in \Omega(f)$$

Or equivalently,

$$g \in \Theta(f) : \exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n)$$

As we discussed, when we write $g \in \Theta(f)$, what we mean is “$g$ grows at most as quickly as $f$ and $g$ grows at least as quickly as $f$”—in other words, that $f$ and $g$ have the same rate of growth. In this case, we call $f$ a tight bound on $g$, since $g$ is essentially squeezed between constant multiples of $f$.

Prove that for all functions $g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$, and all numbers $a \in \mathbb{R}_{\geq 0}$, if $g \in \Omega(1)$, then $a + g \in \Theta(g)$.

[Or in other words, for such functions $g$, shifting them by a constant amount does not change their “Theta” bound.]

**Solution**

Proof. Let $g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$, and let $a \in \mathbb{R}_{\geq 0}$. Assume that $g \in \Omega(1)$, i.e., that there exist $c_0, n_0 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_0$ then $g(n) \geq c_0$. We want to prove that $a + g \in \Theta(g)$, i.e., that there exist $c_1, c_2, n_1 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_1$ then $c_1 g(n) \leq a + g(n) \leq c_2 g(n)$.

Let $c_1 = 1$, $c_2 = \frac{a}{c_0} + 1$, and $n_1 = n_0$. Let $n \in \mathbb{N}$, and assume that $n \geq n_1$. We want to prove that $c_1 g(n) \leq a + g(n) \leq c_2 g(n)$.

[We leave the calculation as an exercise. The trickiest part was figuring out how to choose $c_0$; the intuition is that we need to take the assumed inequality $g(n) \geq c_0$ and turn the right-hand side into $a$ instead of $c_0$.]  

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2 Here we use $a + g$ to denote the function $g_1$ defined as $g_1(n) = a + g(n)$ for all $n \in \mathbb{N}$. 

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4. **Negating Big-Oh.** So far, we have only looked at proving that a function is Big-Oh of another function. In this question, we’ll investigate what it means to show that a function isn’t Big-Oh of another.

(a) Express the statement $g \notin O(f)$ in predicate logic, using the expanded definition of Big-Oh. (As usual, simplify so that all negations are pushed as far “inside” as possible.)

**Solution**

$$g \notin O(f) : \forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \land g(n) > cf(n)$$

(b) Prove that for all positive real numbers $a$ and $b$, if $a > b$ then $n^a \notin O(n^b)$.

**Hint:** for all positive real numbers $x$ and $y$, $x > y \Leftrightarrow \log x > \log y$.

**Solution**

**Proof.** Let $a, b \in \mathbb{R}^+$, and assume that $a > b$. We want to show the following:

$$\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \land n^a > cn^b$$

Let $c, n_0 \in \mathbb{R}^+$. Let $n = \left\lceil n_0 + c^{1/(a-b)} \right\rceil$.

We want to prove that $n \geq n_0$ and $n^a > cn^b$.

[We leave the rest of the proof as an exercise.]

\[ \text{The ceiling function in the choice of } n \text{ is used to ensure that } n \text{ is a natural number. We chose this value of } n \text{ because we want to ensure that } n \geq n_0, \text{ and that } n \geq c^{1/(a-b)}. \]