CSC165 fall 2019

Mathematical expression: induction

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Using Course notes: Induction

$2n + 1 < 2^n$ | false for $n = 0, 1, 2$ | seems true for $n \geq 3$

$P(n): n \geq 3 \Rightarrow 2n+1 < 2^n$, for $\in \mathbb{N}$

Claim $\forall n \in \mathbb{N}, P(n)$. Prove by induction on n

base case, $P(3)$: $2(3)+1 = 7 < 8 = 2^3$, so $P(3)$ is verified.

inductive step Let $n \in \mathbb{N}$. Assume $n \geq 3 \wedge P(n)$, that is $2n+1 < 2^n$. WTS $P(n+1)$, i.e. $2(n+1)+1 < 2^{n+1}$.

IH

$$
\begin{aligned}
2(n+1)+1 &= 2n+2+1 = 2n+1+2 \\
&< 2^n + 2 \quad \# \text{ by IH} \\
&< 2^n + 2^n \quad \# \, n \geq 3 > 1 \\
&= 2^{n+1} \quad \blacksquare
\end{aligned}
$$

$3^n \geq n^3$ $P(n)$: $3^n \geq n^3$

We do inductive step first... you'll see why...

<u>Inductive step</u> Let $n \in \mathbb{N}$. assume $P(n) \leftarrow IH$

That is $3^n \geq n^3$. <u>WTS</u> $P(n+1)$, $3^{n+1} > (n+1)^3$.

$$3^{n+1} = 3 \times 3^n \geq 3 \times n^3 \quad \text{\# by IH.}$$

compare these

$$= n^3 + n^3 + n^3$$

$$\geq n^3 + 3n^2 + 9n \quad \text{\# since } n \geq 3$$

$$= n^3 + 3n^2 + 3n + 6n \quad \text{\# } n \geq 3 \geq \tfrac{1}{6}$$

$$\geq n^3 + 3n^2 + 3n + 1$$

$$= (n+1)^3 \blacksquare$$

base case $\rightarrow$

$$3^n \geq n^3$$

$P(0)$: $3^0 = 1 \geq 0 = 0^3$ ✓

$P(1)$: $3^1 = 3 \geq 1 = 1^3$ ✓

$P(2)$: $3^2 = 9 \geq 8 = 2^3$ ✓

these are true, but do not help inductive step.
Also, $P(i)$ true when i is negative integer.

base case, $P(3)$: $3^3 = 27 \geq 27 = 3^3$ ✓, So $P(3)$ is verified.

$$\forall x, y \in \mathbb{N}, \forall n \in \mathbb{N}, x - y \mid x^n - y^n$$

order of introductions... we introduce arbitrary, fixed pair (x, y), then prove claim for all n for that (x, y). In the other order, for each n prove for all x, y.

Let $x, y \in \mathbb{N}$. Define $P(n)$: $x - y \mid x^n - y^n$, for $n \in \mathbb{N}$.
Prove: $\forall n \in \mathbb{N}, P(n)$, by induction.

base case, $P(0)$: $x^0 - y^0 = 1 - 1 = 0 = 0 \cdot (x - y)$, so $P(0)$ is verified.

Inductive step: Let $n \in \mathbb{N}$. Assume $P(n)$, $x - y \mid x^n - y^n$.
That is, $\exists k_1 \in \mathbb{Z}$, $k_1(x - y) = x^n - y^n$. Let
$k_2 =$ _______________. WTS $k_2(x - y) = x^{n+1} - y^{n+1}$

(next page) →

$x - y \mid x^n - y^n$

(continued)

$$x^{n+1} - y^{n+1} = x(x^n - y^n) + y^n(x-y)$$

$$= x\,k_1(x-y) + y^n(x-y)$$

by IH

$$= (xk_1 + y^n)(x-y)$$

$$= k_2(x-y) \quad \blacksquare$$