test #1 will be returned during next ≈ 22 h
you (royal you) did very well ☺

CSC165 fall 2019

Mathematical expression: induction

Danny Heap
csc165-2019-09@cs.toronto.edu
BA4270 (behind elevators)
Web page:
http://www.teach.cs.toronto.edu/~heap/165/F19/

Using Course notes: Induction

Computer Science
UNIVERSITY OF TORONTO

induction format

for convenience, define a predicate

e.g. $P(n)$: ——————, for $n \in \mathbb{N}$

do not quantify n in def.

- predicate
- base case(s) verify $P(0)$
- inductive step ~ $(\forall n, (P(n) \Rightarrow P(n+1)))$

– you can use $n \to K$

$\forall n, P(n)$ do not confuse, because

$P(n)$: $n = n+1$ (!), for $n \in \mathbb{N}$

$\forall n, P(n) \Rightarrow P(n+1)$

base case fails

prove $\forall n \in \mathbb{N}, 7^n \equiv 1 \pmod 6$

$P(n)$: $\exists k \in \mathbb{Z}, 6k = 7^n - 1$, for $n \in \mathbb{N}$.

Proof that $\forall n \in \mathbb{N}, P(n)$, by induction

base case, $P(0)$: $7^0 - 1 = 1 - 1 = 0 = 6 \cdot 0$,

so $P(0)$ is verified.

inductive step Let $n \in \mathbb{N}$. Assume $P(n)$, (I.H.)

$\exists k_1 \in \mathbb{Z}, 7^n - 1 = 6k_1$. Let $k_2 = 7k_1 + 1$.

wts $6k_2 = 7^{n+1} - 1$.

$$
\begin{aligned}
7^{n+1} - 1 &= 7(7^n - 1) + 6 \\
&= 7 \cdot 6 \cdot k_1 + 6 \quad \# \text{ by IH} \\
&= 6(7k_1 + 1) \\
&= 6k_2 \quad \blacksquare
\end{aligned}
$$

discover, then prove sum of first n numbers result

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \end{array}$$

$$(n+1) + (n+1) + (n+1) + \cdots + (n+1) = \frac{n(n+1)}{2}$$

$P(n)$: $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$, for $n \in \mathbb{N}$.

Proof that $\forall n \in \mathbb{N}, P(n)$, using induction

base case, $P(0)$: $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$, so $P(0)$ is verified.

inductive step: Let $n \in \mathbb{N}$. Assume $P(n)$, that is $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$. WTS $P(n+1)$, that is

$$\sum_{i=0}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

$$\sum_{i=0}^{n+1} = \left[\sum_{i=0}^{n} i\right] + n+1$$

by IH

$$= \frac{n(n+1)}{2} + n+1$$

$$= \frac{n(n+1)+2n+2}{2}$$

$$= \frac{(n+1)(n+2)}{2} \quad ■$$

discover, prove sum of first n cubes result

$0^3 = 0$ $\quad 0^3 + 1^3 = 1$ $\quad 0^3 + 1^3 + 2^3 = 9$

$0^3 + 1^3 + 2^3 + 3^3 = 36$ $\quad 0^3 + 1^3 + 2^3 + 3^3 + 4^3 = 100$

$P(n): \sum_{i=0}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$, for $n \in \mathbb{N}$

Proof that $\forall n \in \mathbb{N}, P(n)$, by induction

base case, $P(0)$ $\quad \sum_{i=0}^{0} i^3 = 0 = \left[\frac{0(0+1)}{2}\right]^2$, so

$P(0)$ verified.

inductive step. Let $n \in \mathbb{N}$. Assume the IH $P(n)$,

$\sum_{i=0}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$. WTS $\sum_{i=0}^{n+1} i^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2$.

$\sum_{i=0}^{n+1} i^3 = \left[\sum_{i=0}^{n} i^3\right] + (n+1)^3$ ← exercise!