Learning Objectives

By the end of this worksheet, you will:

- Prove and disprove statements about numbers and functions.
- Use mathematical definitions of predicates to simplify or expand formulas.
- Identify errors in an incorrect proof.

1. A direct proof. Recall that we say an integer $n$ is odd if and only if $\exists k \in \mathbb{Z}, \ n = 2k - 1$. Using the technique from lecture, prove the following statement:

   For every pair of odd integers, their product is odd.

   Be sure to translate the statement into predicate logic. You can use the predicate $Odd(n)$: “$n$ is odd” in your formula without expanding the definition, but you’ll need to use the definition in your proof.

2. An incorrect proof. Consider the following claim:

   For every even integer $m$ and odd integer $n$, $m^2 - n^2 = m + n$.

   (a) Using the predicates $Even(n)$ and $Odd(n)$ (which return whether an integer $n$ is even or odd, respectively), express the above statement using the notation of symbolic logic.

   (b) The following argument was submitted as a proof of the statement:

   Proof. Let $m$ and $n$ be arbitrary integers, and assume $m$ is even and $n$ is odd. By the definition of even, $\exists k \in \mathbb{Z}, m = 2k$; by the definition of odd, $\exists k \in \mathbb{Z}, n = 2k - 1$. We can then perform the following algebraic manipulations:

   $$m^2 - n^2 = (2k)^2 - (2k - 1)^2$$
   $$= 4k^2 - 4k^2 + 4k - 1$$
   $$= 4k - 1$$
   $$= 2k + (2k - 1)$$
   $$= m + n$$
The given argument is not a correct proof. What is the flaw?\[1\]

3. **Comparing functions.** Consider the following definition\[2\]

**Definition 1.** Let \( f, g : \mathbb{N} \rightarrow \mathbb{R}^0 \). We say that \( g \) is dominated by \( f \) (or \( f \) dominates \( g \)) if and only if for every natural number \( n \), \( g(n) \leq f(n) \).

(a) Express this definition symbolically by showing how to define the following predicate:

\[
\text{Dom}(f, g) : \quad \text{______________________________, where } f, g : \mathbb{N} \rightarrow \mathbb{R}^0.
\]

(b) Let \( f(n) = 3n \) and \( g(n) = n \). Prove that \( g \) is dominated by \( f \).

(c) Let \( f(n) = n^2 \) and \( g(n) = n + 165 \). Prove that \( g \) is not dominated by \( f \). Make sure to write the statement.

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\[1\] If you have time, you might want to consider whether the given statement is True or False, and write a correct proof or disproof.

\[2\] We'll use the symbol \( \mathbb{R}^0 \) to denote the set of all nonnegative real numbers, i.e., \( \mathbb{R}^0 = \{ x \mid x \in \mathbb{R} \land x \geq 0 \} \).
you’ll prove in predicate logic, in fully simplified form (negations moved all the way inside).

(d) Now let’s generalize the previous statement. Translate the following statement into predicate logic (expanding the definition of $\text{Dom}$) and then prove it!

For every positive real number $x$, $g(n) = n + x$ is not dominated by $f(n) = n^2$.

4. More with floor. Recall that the floor of a number $x$, denoted $\lfloor x \rfloor$, is the largest integer less than or equal to $x$. For every $x \in \mathbb{R}$, there exists an $\epsilon \in \mathbb{R}$ such that $x = \lfloor x \rfloor + \epsilon$, and $0 \leq \epsilon < 1$.

Prove the following statement:\footnote{For extra practice, try proving the following generalization of this statement: $\forall k \in \mathbb{R}^{\geq 0}, \ k < 1 \Rightarrow (\exists x_0 \in \mathbb{R}^{\geq 0}, \ \forall x \in \mathbb{R}^{\geq 0}, \ x \geq x_0 \Rightarrow (\lfloor x \rfloor)^2 \geq kx^2)$.}

$$\forall x \in \mathbb{R}^{\geq 0}, \ x \geq 4 \Rightarrow (\lfloor x \rfloor)^2 \geq \frac{1}{2}x^2$$

Hint: First introduce a variable $\epsilon$ and rewrite $\lfloor x \rfloor$ as $x - \epsilon$. Then, prove the following simpler statement, and use it in your proof: $\forall x \in \mathbb{R}^{\geq 0}, \ x \geq 4 \Rightarrow \frac{1}{2}x^2 \geq 2x$. 
