- just before lecture I am distracted by my tech...
- ignore the sign + bring food + coffee to lecture, as needed

CSC165 fall 2019

Mathematical expression: predicate logic

- if you know somebody enrolled but lukewarm about the course, urge them to give up their place to somebody on waiting list...

Danny Heap

csc165-2019-09@cs.toronto.edu

BA4270 (behind elevators)

Web page:

http://www.teach.cs.toronto.edu/~heap/165/F19/

Using Course notes: Mathematical Expression: predicate logic

Outline

implication with predicates

quantifiers

multiple quantifiers

negation

number theory intro

notes

the venn of implication...

This works in the situation where an implication is true for the entire domain

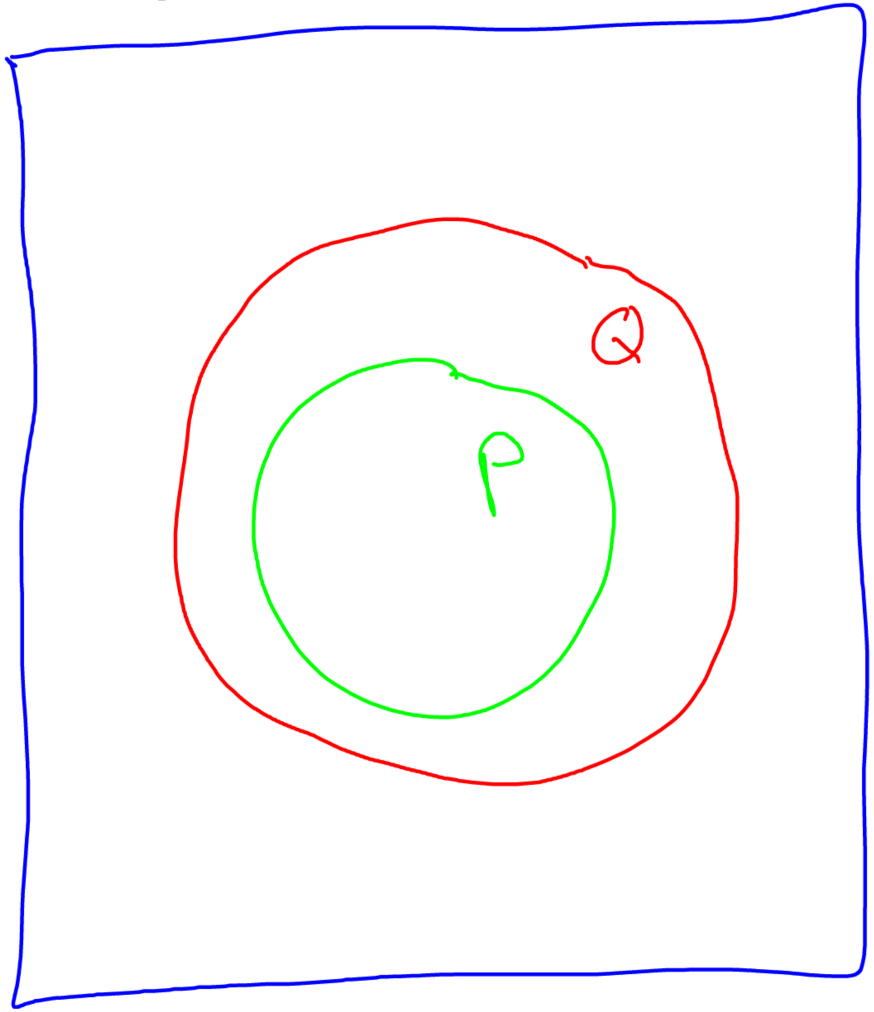
$p(x)$: "$x > 4$", for $x \in \mathbb{R}$

$q(x)$: "$x > 3$", for $x \in \mathbb{R}$

$f(x)$ "$x^2 + 1 = 0$", for $x \in \mathbb{R}$

define $P = \{x : x \in \mathbb{R} \wedge p(x)\}$, $Q = \{x : x \in \mathbb{R} \wedge q(x)\}$, $F = \{x : x \in \mathbb{R} \wedge f(x)\}$

draw Venn diagrams of P, Q, F and compare to $p(x) \Rightarrow q(x), f(x) \Rightarrow q(x)$



$\mathbb{R}$

note $|F| = 0$, i.e. $F = \emptyset$

$p(x) \Rightarrow q(x)$

equivalent

$P \subseteq Q$

similarly $f(x) \Rightarrow q(x)$

equivalent $F \subseteq Q$

more venn of implication...

again, a situation where implications are true for *entire* domain

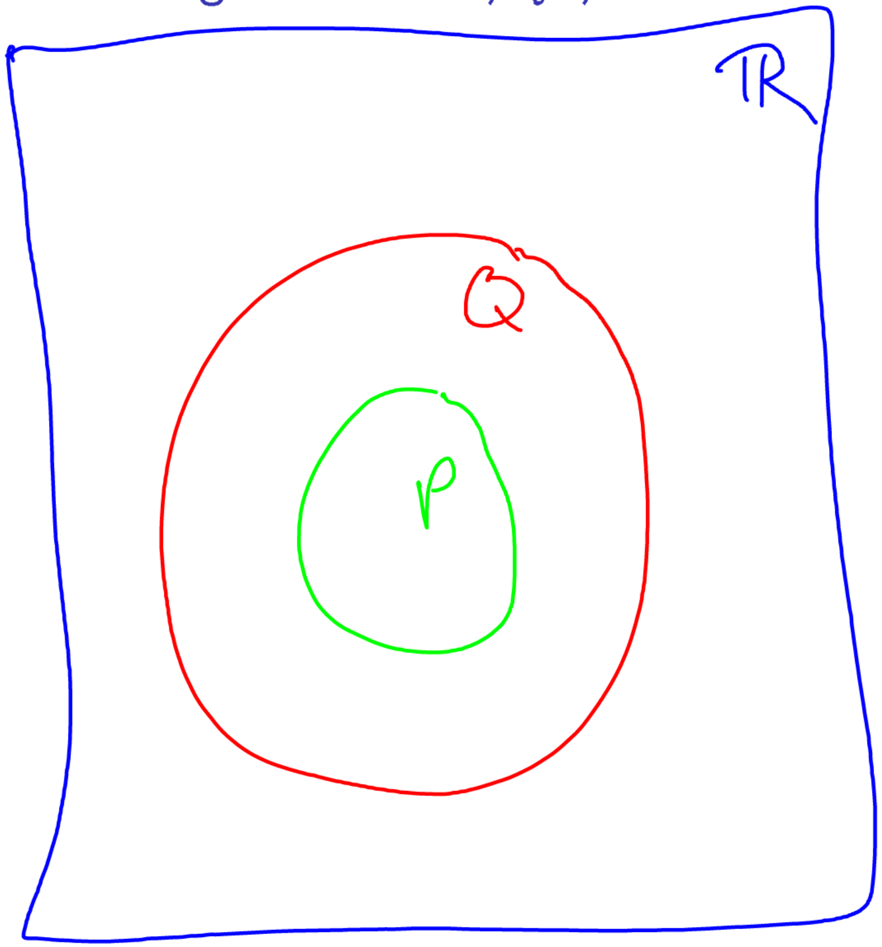
$p(x)$: "$x > 4$", for $x \in \mathbb{R}$

$q(x)$: "$x > 3$", for $x \in \mathbb{R}$

$f(x)$ "$x^2 + 1 = 0$", for $x \in \mathbb{R}$

define $P = \{x : x \in \mathbb{R} \wedge p(x)\}$, $Q = \{x : x \in \mathbb{R} \wedge q(x)\}$, $F = \{x : x \in \mathbb{R} \wedge f(x)\}$

draw Venn diagrams of P^c, Q^c, F^c and compare to $\neg q(x) \Rightarrow \neg p(x), \neg q(x) \Rightarrow \neg f(x)$



$\neg q(x) \Rightarrow \neg p(x)$

equivalent $Q^c \subseteq P^c$

also

$\neg q(x) \Rightarrow \neg f(x)$

equivalent $Q^c \subseteq F^c$

$F^c = \mathbb{R}$

quantifiers $\forall$ and $\exists$

$f(n)$: "$n > 7$", for $n \in \mathbb{N}$ — predicate output depends on n...

universal quantifier $\forall$

$\forall n \in \mathbb{N}, n > 7$ means

$0>7 \wedge 1>7 \wedge 2>7 \wedge \ldots \wedge 8>7 \wedge 9>7 \wedge \ldots$

False, eg $0>7$

existential quantifier $\exists$

$\exists n \in \mathbb{N}, n > 7$ means

$0>7 \vee 1>7 \vee 2>7 \vee \ldots \vee 8>7 \vee 9>7 \vee \ldots$

True e.g. $9>7$

translate quantified predicates

$\forall n \in \mathbb{N}, n > 7$

- "For all natural numbers n, n is greater than 7."
- "Every natural number is greater than 7."

$\exists n \in \mathbb{N}, n > 7$

- "There exists a natural number n such that n is greater than 7."
- "Some natural number is greater than 7."

multiple matched quantifiers...

$A = \{1, 3, 5\}$, $B = \{6, 8, 10\}$

$p(x, y)$: "$x + y = 11$", for $(x, y) \in A \times B$

$\forall x \in A, \forall y \in B, p(x, y)$ versus $\exists x \in A, \exists y \in B, p(x, y)$

step 1: conjunction using each element of A

$(\forall y \in B, 1+y=11) \wedge (\forall y \in B, 3+y=11) \wedge (\forall y \in B, 5+y=11)$

Step 2: conjunction using each element of B

$(1+6=11 \wedge 1+8=11 \wedge 1+10=11) \wedge (3+6=11 \wedge 3+8=11 \wedge 3+10=11)$
$\wedge (5+6=11 \wedge 5+8=11 \wedge 5+10=11)$

False! Since all operators are $\wedge$, we may regroup:

$(1+6=11 \wedge 3+6=11 \wedge 5+6=11) \wedge (1+8=11 \wedge 3+8=11 \wedge 5+8=11)$
$\wedge (1+10=11 \wedge 3+10=11 \wedge 5+10=11) \rightarrow \forall y \in B, \forall x \in A, p(x,y)$

order doesn't matter!

multiple matched quantifiers continued...

$A = \{1,3,5\}$, $B = \{6,8,10\}$

$p(x,y)$: "$x + y = 11$", for $(x,y) \in A \times B$

$\forall x \in A, \forall y \in B, p(x,y)$ versus $\boxed{\exists x \in A, \exists y \in B, p(x,y)}$

Step 1: disjunction over all elements of A

$(\exists y \in B, 1+y=11) \vee (\exists y \in B, 3+y=11) \vee (\exists y \in B, 5+y=11)$

step 2: disjunction over all elements of B

$(1+6=11 \vee 1+8=11 \vee 1+10=11) \vee (3+6=11 \vee 3+8=11 \vee 3+10=11)$
$\vee (5+6=11 \vee 5+8=11 \vee 5+10=11)$

True also, since all operators are $\vee$, we may re-group:

$(1+6=11 \vee 3+6=11 \vee 5+6=11) \vee (1+8=11 \vee 3+8=11 \vee 5+8=11)$
$\vee (1+10=11 \vee 3+10=11 \vee 5+10=11)$

$\exists y \in B, \exists x \in A, p(x,y)$

order doesn't matter here!

mixed quantifiers

$A = \{1, 3, 5\}$, $B = \{6, 8, 10\}$

$p(x, y)$: "$x + y = 11$", for $(x, y) \in A \times B$

$\forall x \in A, \exists y \in B, p(x, y)$ versus $\exists y \in B, \forall x \in A, p(x, y)$

step 1: conjunction over all elements of A

$(\exists y \in B, 1+y=11) \wedge (\exists y \in B, 3+y=11) \wedge (\exists y \in B, 5+y=11)$

step 2: disjunction over all elements of B

$(1+6=11 \vee 1+8=11 \vee 1+10=11) \wedge (3+6=11 \vee 3+8=11 \vee 3+10=11)$.

$\wedge (5+6=11 \vee 5+8=11 \vee 5+10=11)$

True!

different from next slide

Also, cannot regroup $\wedge, \vee$ so... order matters... see next slide

mixed quantifiers continued...

$A = \{1, 3, 5\}$, $B = \{6, 8, 10\}$

$p(x, y)$: "$x + y = 11$", for $(x, y) \in A \times B$

$\forall x \in A, \exists y \in B, p(x, y)$ versus $\exists y \in B, \forall x \in A, p(x, y)$

step 1: disjunction over all elements of B

$(\forall x \in A, x + 6 = 11) \vee (\forall x \in A, x + 8 = 11) \vee (\forall x \in A, x + 10 = 11)$

step 2: conjunction over all elements of A

$(1+6 = 11 \wedge 3+6 = 11 \wedge 5+6 = 11) \vee (1+8 = 11 \wedge 3+8 = 11 \wedge 5+8 = 11)$
$\vee (1+10 = 11 \wedge 3+10 = 11 \wedge 5+10 = 11)$

False! Also, cannot regroup $\vee, \wedge$ so... order matters

different from previous!

$\exists$: examples

$\forall$: lack of counterexamples

negate quantified predicates

manipulate negation

$$\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, x \geq 5 \vee x^2 - y \geq 30$$

properties of integers, mostly $\mathbb{N}$

divisibility

primes

Notes