Learning Objectives

By the end of this worksheet, you will:

- Analyse the average-case running time of an algorithm.

1. Average-case analysis. Consider the following algorithm that we studied a few weeks ago. The input is an array $A$ of length $n$, containing a list of $n$ integers.

```python
def has_even(A: List[int]) -> int:
    n = len(A)
    for i in range(n):
        if A[i] % 2 == 0:
            return i
    return -1
```

We proved that the worst-case running time of this algorithm is $\Theta(n)$. In this problem we will analyse its average-case running time.

For this analysis, we will consider the set of binary lists $A$ of length $n$. That is, $A$ is a list of $n$ integers, where each integer is either 0 or 1.

(a) For each $n \in \mathbb{Z}^+$, let $I_n$ be the set of all binary lists of length $n$. Find an expression (in terms of $n$) for $|I_n|$, the size of $I_n$.

**Solution**

The number of binary lists of length $n$ is $2^n$, thus $|I_n| = 2^n$. 

(b) For each $n \in \mathbb{Z}^+$ and each $i \in \{0, 1, \ldots, n-1\}$, let $S_n(i)$ denote the set of all binary lists $A$ such that the first 0 occurs in position $i$. More precisely, every list in $S_n(i)$ satisfies the following two properties:

(i) $A[i] = 0$.

(ii) for all $j \in \mathbb{N}$, if $j < i$ then $A[j] = 1$.

Also let $S_n(n)$ be the set of binary lists that contain no 0's. For each $i$, $0 \leq i \leq n$, find an expression for $|S_n(i)|$.

**Solution**

For $0 \leq i \leq n - 1$, $|S_n(i)| = 2^{n-1-i}$.

Also, $|S_n(n)| = 1$.

(c) Argue that for every $n \in \mathbb{Z}^+$, each binary list of length $n$ is in exactly one set $S_n(i)$ (for some $i \in \{0, \ldots, n\}$). Use this to show that $\sum_{i=0}^{n} |S_n(i)| = |Z_n|$.

**Solution**

For each input, either it contains a 0 or it doesn’t. If it doesn’t then it is (the single input) in $S_n(n)$. If it does, then we partition these inputs according to the smallest location $i \leq n - 1$ where $A[i] = 0$: if an input has its first 0 in $A[i]$, then it is in the set $S_n(i)$. The sum is $2^{n-1} + 2^{n-2} + \ldots + 1 + 1 = 2^n$. 
(d) Let the runtime of the algorithm on a binary list $A$ be the number of iterations of the loop. Give an exact expression for the average runtime of the above algorithm using the quantities that you calculated. You should get a summation; do not simplify the summation in this part.

**Solution**

Note that each input in $S_n(i)$ causes the loop to execute exactly $i + 1$ times. So the overall average runtime is:

$$\frac{1}{2^n} \sum_{i=0}^{n} |S_n(i)| \times (i + 1) = \left( \frac{1}{2^n} \sum_{i=0}^{n-1} |S_n(i)| \times (i + 1) \right) + \frac{|S_n(n)| \times (n + 1)}{2^n}$$

$$= \left( \frac{1}{2^n} \sum_{i=0}^{n-1} 2^{n-1-i} \times (i + 1) \right) + \frac{n + 1}{2^n}$$

$$= \left( \frac{1}{2^n} \sum_{i'=1}^{n} 2^{n-i'} \times i' \right) + \frac{n + 1}{2^n} \quad \text{(change of variable $i' = i + 1$)}$$

$$= \left( \sum_{i'=1}^{n} \left( \frac{1}{2} \right)^{i'} \times i' \right) + \frac{n + 1}{2^n}$$

(e) Show that the runtime that you calculated is in $O(1)$. You may use without proof that for all $x \in \mathbb{R}$, if $|x| < 1$, then $\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$.

**Solution**

So we have $(n + 1)/2^n + \sum_{i'=1}^{n} i'(1/2)^{i'}$. The first part is eventually less than 1, and by the formula given above, the second part is at most 2. Thus the expected runtime is $O(1)$. 