Learning Objectives

By the end of this worksheet, you will:

- Analyse the average-case running time of an algorithm.

1. **Average-case analysis.** Consider the following algorithm that we studied a few weeks ago. The input is an array $A$ of length $n$, containing a list of $n$ integers.

```python
def has_even(A: List[int]) -> int:
    n = len(A)
    for i in range(n):
        if A[i] % 2 == 0:
            return i
    return -1
```

We proved that the worst-case running time of this algorithm is $\Theta(n)$. In this problem we will analyse its *average-case* running time.

For this analysis, we will consider the set of binary lists $A$ of length $n$. That is, $A$ is a list of $n$ integers, where each integer is either 0 or 1.

(a) For each $n \in \mathbb{Z}^+$, let $I_n$ be the set of all binary lists of length $n$. Find an expression (in terms of $n$) for $|I_n|$, the size of $I_n$. 

\[ I_n = \text{set of all binary lists of length } n \]
(b) For each $n \in \mathbb{Z}^+$ and each $i \in \{0, 1, \ldots, n-1\}$, let $S_n(i)$ denote the set of all binary lists $A$ such that the first 0 occurs in position $i$. More precisely, every list in $S_n(i)$ satisfies the following two properties:

(i) $A[i] = 0$.

(ii) for all $j \in \mathbb{N}$, if $j < i$ then $A[j] = 1$.

Also let $S_n(n)$ be the set of binary lists that contain no 0's. For each $i$, $0 \leq i \leq n$, find an expression for $|S_n(i)|$.

(c) Argue that for every $n \in \mathbb{Z}^+$, each binary list of length $n$ is in exactly one set $S_n(i)$ (for some $i \in \{0, \ldots, n\}$).

Use this to show that $\sum_{i=0}^{n} |S_n(i)| = |\mathbb{Z}_n|$. 
(d) Let the runtime of the algorithm on a binary list $A$ be the number of iterations of the loop. Give an exact expression for the average runtime of the above algorithm using the quantities that you calculated. You should get a summation; do not simplify the summation in this part.

(e) Show that the runtime that you calculated is in $O(1)$. You may use without proof that for all $x \in \mathbb{R}$, if $|x| < 1$, then $\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$. 