test 2 may include: induction, binary representation, big Oh/Omega/Theta worst-case, average case, exact analysis.

CSC165 fall 2019

average/summation/graphs

ps4 up by (end of?) weekend...

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Web page:

http://www.teach.cs.toronto.edu/~heap/165/F19/

Using Course notes: average analysis; graphs

average...

$$\text{average} = \sum_{x \in I_{has\_even,n}} RT_{has\_even}(x) \Big/ |I_{has\_even,n}|$$

simplify number_list $\in \{0, 1\}$

$$\mathcal{I}_{f,n} = \{i \mid i \text{ is an input to } f \wedge |i| = n\}$$

$n = 2$

1 step, 2 lists $\longrightarrow$ 2
2 steps, 1 list $\longrightarrow$ 2
3 steps, 1 list $\longrightarrow$ 3

$$\frac{\sum RT = 7}{4}$$

```
def has_even(number_list):
    for number in number_list:
        if number % 2 == 0:
            return True
    return False
```

average...

$n=3$

[0,0,1]
[0,1,0]

1 step, 4 lists → 4
2 steps, 2 lists → 4
3 steps, 1 list → 3
4 steps, 1 → 4

$$\mathcal{I}_{f,n} = \{i \mid i \text{ is an input to } f \wedge |i| = n\}$$

$$\frac{\sum RT = 15}{2^3 = 8}$$

```
def has_even(number_list):
    for number in number_list:
        if number % 2 == 0:
            return True
    return False
```

average...

$n = 4$

1 step, 8 lists $\longrightarrow$ 8

2 steps, 4 lists $\longrightarrow$ 8

3 steps, 2 lists $\longrightarrow$ 6

4 steps, 1 list $\longrightarrow$ 4

5 steps, 1 list $\longrightarrow$ 5

$\sum RT = 31$

$2^4 = 16$

$$\mathcal{I}_{f,n} = \{ i \mid i \text{ is an input to } f \land |i| = n \}$$

```
def has_even(number_list):
    for number in number_list:
        if number % 2 == 0:
            return True
    return False
```

observations

for $I_{h\_e,n}$, 2^n lists

steps

$$\underbrace{1 \cdot 2^{n-1} + 2 \cdot 2^{n-2} + 3 \cdot 2^{n-3} + \cdots + n \cdot 2^{n-n} + n+1}_{2^n}$$

summation...

from notes...

$$\sum_{i=0}^{i=n-1} ir^i = \frac{nr^n}{r-1} + \frac{r-r^{n+1}}{(r-1)^2}$$

want to evaluate

$$\frac{\left(\sum_{i=1}^{n} i2^{n-i}\right) + n+1}{2^n} = \frac{\left(2^n \sum_{i=1}^{n} i2^{-i}\right) + n+1}{2^n}$$

$$= \left(\sum_{i=1}^{n} i2^{-i}\right) + \frac{n+1}{2^n}$$

summation...

from notes...

$$\sum_{i=0}^{i=n-1} ir^i = \frac{nr^n}{r-1} + \frac{r - r^{n+1}}{(r-1)^2}$$

$$= \left(\sum_{i=1}^{n} i 2^{-i} \right) + \frac{n+1}{2^n}$$

$$= \left(\sum_{i=1}^{n} i \left(\tfrac{1}{2}\right)^i \right) + \frac{n+1}{2^n} = \left(\sum_{i=0}^{n} i \left(\tfrac{1}{2}\right)^i \right) + \frac{n+1}{2^n}$$

$$= (\text{use formula, } n \to n+1) = \frac{-(n+1)\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)^1} + \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{n+2}}{\left(\frac{1}{2}\right)^2}$$

$$= -(n+1)\left(\tfrac{1}{2}\right)^n + 2 - \left(\tfrac{1}{2}\right)^n + \frac{n+1}{2^n}$$

$$= 2 - (n+2)/2^n + \frac{n+1}{2^n} = 2 - \frac{1}{2^n}$$

$$\sum_{i=0}^{n-1} i r^i = \sum_{j=1}^{n} (j-1) r^{j-1}$$

$$= \sum_{j=1}^{n} j r^{j-1} - \sum_{j=1}^{n} r^{j-1}$$

$$= \frac{1}{r} \sum_{j=1}^{n} j r^j - \frac{1}{r} \sum_{j=1}^{n} r^j$$

advice write out in long

$+ \cdots + \cdots +$

subtract from both sides

add to both sides

move out last term

$$\frac{1}{r} \sum_{j=1}^{n} r^j = \frac{1}{r} \sum_{j=0}^{n} j r^j - \sum_{i=0}^{n-1} i r^i$$

$$= \left(\frac{1}{r} - 1\right) \sum_{i=0}^{n-1} i r^i + \frac{1}{r} \cdot n \cdot r^n$$

$$\frac{1}{r} \sum_{j=1}^{n} r^j - \frac{1 \cdot n}{r} \cdot r^n = \left(\frac{1}{r} - 1\right) \sum_{i=0}^{n-1} i r^i \longrightarrow$$

$$\frac{1}{r}\sum_{j=1}^{n} r^j - \frac{1\cdot n \cdot r^n}{r} = \left(\frac{1}{r}-1\right)\sum_{i=0}^{n-1} i r^i$$

$$\frac{\frac{1}{r}\sum_{j=1}^{n} r^j - \frac{nr^n}{r}}{\frac{1}{r}-1} = \sum_{i=0}^{n-1} i r^i$$

$$\frac{\frac{\sum_{j=1}^{n} r^j}{r} - \frac{nr^n}{r}}{\frac{1-r}{r}} = \frac{\overbrace{\sum_{j=1}^{n} r^j}^{\text{geometric series } = \frac{r^{n+1}-r}{r-1}} - nr^n}{1-r} = \sum_{i=0}^{n-1} i r^i$$

$$= \frac{1}{1-r}\left[\frac{r^{n+1}-r}{r-1} - nr^n\right] = (-1)\frac{1}{r-1}\left[\frac{r^{n+1}-r}{r-1} - nr^n\right]$$

$$= \frac{nr^n}{r-1} - \frac{r^{n+1}-r}{(r-1)^2} \quad \text{(whew!)}$$