CSC165H1: Problem Set 4 Sample Solutions

Due December 4, 2019 before 4pm

Note: solutions are incomplete, and meant to be used as guidelines only. We encourage you to ask follow-up questions on the course forum or during office hours.

1. **[12 marks] algorithm analysis**

Read over the code for **has_odd**. Assume that **number_list** contains entries from $\{0, 1, 2, 3, 4\}$, with duplicates allowed. Answer the questions below, assuming that $n = \textbf{len}(\text{number\_list})$

```
def has_odd(number_list) -> bool:
  for i in range(len(number_list)):
    if number_list[i] % 2 == 1:
      return True
  return False
```

(a) **[3 marks]** Find a good upper bound, $U(n)$, for $WC_{has\_odd}(n)$. Prove that your upper bound is correct.

Solution

sample solution:

header: Let $n \in \mathbb{N}$. Let **UL** be an arbitrary list of n numbers from $\{0, 1, 2, 3, 4\}$. I want to show that **has_odd** takes no more than $U(n) = n + 1$ steps.

body: I assign 1 step to the body of the loop. This means that **has_odd** takes no more than n steps to examine every member of list **UL**, and no more than 1 more step to return **False**. In all, no more than $U(n) = n + 1$ for any list of length n, including a list that generates the worst case. steps. ■

(b) **[3 marks]** Find a lower bound, $L(n)$, for $WC_{has\_odd}(n)$ that is in the same asymptotic complexity class as $U(n)$ (that's what I mean by "good" in the previous part). Prove that your lower bound is correct, then state and justify a simple big-Theta complexity class for $WC_{has\_odd}(n)$

Solution

sample solution:

header: Let $n \in \mathbb{N}$. Let **LL** be a list consisting of n 2s. Then **has_odd** examines all n entries in **LL**, at a cost of n steps, and then takes one more step to return **False**. Thus **LL** requires at least $L(n) = n + 1$ steps, and hence so does any list that generates the worst case number of steps.. ■

Both $U(n)$ and $L(n)$ are in $\Theta(n)$.

(c) **[3 marks]** If **has_odd** returns **True** after examining k entries in **number_list**, count this as k steps. If an **has_odd** examines all n entries in **number_list** and proceeds to return **False** count this as $n + 1$ steps. Using these assumptions, show how to calculate the average number of steps for all inputs to **has_odd** of length 2.

<u>**Solution**</u>

sample solution: Here are the run-time costs broken down according to how many steps it takes to either return **True** or else return **False**:

- There are 2×5 lists with an odd number in the first position: a choice of 2 in the first entry, times a choice of 5 for the second position. Each of these costs 1 step to return **True**.
- There are 3×2 lists with the first occurrence of an odd number in the second position: a choice of 3 even numbers in the first position, times a choice of 2 odd numbers in the second position. Each of these costs 2 steps to return **True**.
- There are 3×3 lists with no occurrence of an odd number: a choice of 3 even numbers in the first position times a choice of 3 even numbers in the second position. each of these costs 3 steps to return **False**.

This makes the sum of steps over all lists:

$$10 \times 1 + 6 \times 2 + 9 \times 3 = 49$$

Dividing by the total number of lists, 25, gives an average number of steps of:

$$\frac{49}{25}$$

...or a bit less than 2.

(d) **[3 marks]** Use the step-counting assumptions in the previous part to devise a formula for the average number of steps for all inputs to **has_odd** of length n. You may find it useful to recall (where r is some positive real number)

$$\sum_{i=0}^{i=n-1} ir^i = \frac{nr^n}{r-1} + \frac{r - r^{n+1}}{(r-1)^2}$$

Show your work.

<u>**Solution**</u>

sample solution: If the first occurrence of an odd number occurs at position i in an input list, then there are 3^{i-1} possible entries before position i (even numbers) and 5^{n-i} possible entries after position i (all 5 numbers). In position i itself there are 2 possibilities: 1 or 3. Each of these lists requires i steps to process.

In addition, there are 3^n lists that have no odd numbers, and require $n + 1$ steps to process.

The sum of steps must be divided by 5^n, the total number of lists.

$$\begin{aligned}
\frac{1}{5^n}\left(\sum_{i=1}^{i=n} i(3^{i-1})(2)5^{n-i}\right) + \frac{3^n(n+1)}{5^n} &= \frac{2}{3}\left(\sum_{i=1}^{i=n} i(3/5)^i\right) + \frac{3^n(n+1)}{5^n} \\
&\quad \#\text{ factor out } 2, 3^{-1}, \text{ and } 5^n \\
&= \frac{2}{3}\left(\frac{(n+1)(3/5)^{n+1}}{-(2/5)} + \frac{(3/5)-(3/5)^{n+2}}{(4/25)}\right) + \frac{3^n(n+1)}{5^n} \\
&\quad \#\text{ sub } n+1 \text{ for } n \text{ in formula...} \\
&= -(n+1)(3/5)^n + 5/2 - (3/2)(3/5)^n + (n+1)(3/5)^n \\
&= \frac{5-3(3/5)^n}{2}
\end{aligned}$$

2. **[18 marks] graph connectivity**

Answer the questions below. Assume $|V|$ is finite and positive.

(a) **[3 marks]** Prove that for all undirected graphs $G = (V, E)$, if C is a cycle in G and e is an edge in C, then removing e leaves C connected. Notice that this is used in Example 6.8, so you cannot use Example 6.8 as proof, nor can you use the fact that this is asserted without proof in the paragraph just before Example 6.8.

Solution

sample solution:

header: Let $G = (V, E)$ be an undirected graph, and $C = v_0, \ldots, v_{k-1}, v_k = v_0$ be a cycle in G with all vertices distinct except $v_0 = v_k$, and edges (v_i, v_{i+1}) for all $0 \le i < k$. Let $e = (v_j, v_{j+1})$, $0 \le j < k$ be an arbitrary edge in C. Let $C' = C - e$, the cycle after removing e. Let v_h, v_m, $0 \le h < m < k$ be two arbitrary, distinct vertices in C. I want to show that there is a path in C' between v_h and v_m.

body: There are three cases to consider, depending on where v_h and v_m lie with respect to removed edge $e = (v_j, v_{j+1})$.

case $m \le j$: Then $v_h, \ldots, v_m$ is a subsequence of the distinct vertices in C, and $\forall h \le n < m$ edge (v_n, v_{n+1}) is inherited from C and has not been removed, since $m \le j$. This forms a path from v_h to v_m (and vice-versa in an undirected graph).

case $j + 1 \le h$: Then $v_h, \ldots, v_m$ is a subsequence of the distinct vertices in C, and $\forall h \le n < m$ edge (v_n, v_{n+1}) is inherited from C and has not been removed, since $j + 1 \le h$. This forms a path from v_h to v_m (and vice-versa in an undirected graph).

case $h \le j \wedge j + 1 \le m$: Then $v_m, \ldots, v_k = v_0$ is a sequence of vertices inherited from C, with corresponding connecting edges, connecting v_m to v_0. Also $v_0, \ldots, v_h$ is a subsequence of the vertices from C, with corresponding connecting edges connecting v_0 to v_h. By transitivity of connectedness, v_m is connected to v_h (and vice-versa in undirected graphs).

In all three possible cases $C' = C - e$ is connected. ■

(b) **[3 marks]** Prove or disprove: In every undirected graph $G = (V, E)$ with all vertices having degree at least $\lfloor |V|/3 \rfloor$, for every 3 distinct vertices $u, v, w \in V$ there is a path of length no more than 2 from u to v, or from v to w, or from w to u.

Solution

sample solution:

header: Let $G = (V, E)$ be an arbitrary undirected graph. Assume $\forall v \in V, d(v) \geq \lfloor |V|/3 \rfloor$. Let $u, v, w \in V$ be distinct vertices. I will show that either u and v share a neighbour, or that u and w share a neighbour, or that v and w share a neighbour.

body: There are two cases to consider.

case u and v share a neighbour: Then we are done.

case u and v do not shaare neighbour: Let $N(u) = \{u\} \cup \{x : x \in V \wedge (u, x) \in E\}$, $N(v) = \{v\} \cup \{y : y \in V \wedge (u, y) \in E\}$, and $N(w) = \{w\} \cup \{z : z \in V \wedge (u, z) \in E\}$ — the neighbourhods of u, v, and w. Since u and v share no neighbours $N(u)$ and $N(v)$ are disjoint, so

$$\begin{aligned}
|N(u) \cup N(v)| &\geq 2(\lfloor |V|/3 \rfloor + 1) \geq 2((|V| - 2)/3 + 1) \\
& \qquad \# \; \exists q, r \in \mathbb{Z}, \lfloor |V|/3 \rfloor = \lfloor (3q + r)/3 \rfloor = q \wedge 3 > r \geq 0 \\
-|N(u) \cup N(v)| &\leq -2((|V| - 2)/3 + 1) \qquad \# \text{ multiply by } -1 \\
|V| - |N(u) \cup N(v)| &\leq |V| - 2((|V| - 2)/3 + 1) = (|V| - 2)/3 \leq \lfloor |V|/3 \rfloor
\end{aligned}$$

But then there are not enough vertices in $V \setminus (N(u) \cup N(v))$ for all of $N(w)$, so $N(w)$ must share at least one neighbour with either $N(u)$ or $N(v)$. ■

(c) **[3 marks]** Prove or disprove: If graph $G = (V, E)$ has an odd number of vertices with even degree, then $|V|$ is odd.

Solution

sample solution: First I need a lemma: Every graph $G = (V, E)$ has an even number of vertices of odd degree. I prove this by induction on $|E|$, the number of edges.

header: Define $P(m)$:"Every graph $G = (V, E)$ with $|E| = m$ has an even number of vertices of odd degree."

base case $P(0)$: A graph with 0 edges has 0 (an even number) vertices of odd degree, which verifies $P(0)$.

inductive step: Let $m \in \mathbb{N}$ and assume $P(n)$. Let $G = (V, E)$ be an arbitrary graph with $m + 1$ edges. Since $m + 1 > 0$, we can remove one edge, (u, v) creating graph G' with m edges, so by the inductive hypothesis G' has an even number of vertices of odd degree. There are three cases to consider:

case u, v both of odd degree in G': Then G has two fewer vertices of odd degree than G' does. That is, there exists a natural number k so that G' has $2k$ vertices of odd degree, and $k > 0$. Then G has $2(k - 1)$ vertices off odd degree, also an even number.

case u, v both of even degree in G': Then G has two more vertices of odd degree than G' does. That is, there exists a natural number k' so that G' has $2k'$ vertices of odd degree. Then

G has $2(k'+1)$ vertices of odd degree, also an even number.

case one of u, v has odd degree, the other even degree in G': Then the parity of u and v's degree is switched in G versus G', and they have exactly the same number (an even one) of vertices of odd degree.

In all three possible cases G has an even number of vertices of odd degree. ■

main proof: Let $G = (V, E)$ be a graph with an odd number of vertices of even degree, so there is some natural number h such that G has $2h + 1$ vertices of even degree. By the lemma above G has an even number of vertices of odd degree, so there is some natural number i such that G has $2i$ vertices of odd degree. All together G has $2h + 1 + 2i = 2(h + i) + 1$ vertices, an odd number. ■

(d) **[3 marks]** Prove or disprove: Every undirected graph $G = (V, E)$ with at least 13 vertices, all vertices having degree at least $|V| - 7$, is connected.

Solution

sample solution: I will prove the contrapositive:

$$\forall G = (V, E), G \text{ is not connected} \Rightarrow |V| < 13 \vee (\exists v \in V, d(v) < |V| - 7)$$

header: Let $G = (V, E)$ be an arbitrary graph disconnected graph. In other words, there exist $u, v \in V$ that are not connected. Let $N(u) = \{u\} \cup \{w \in V : (u, w) \in E\}$ and $N(v) = \{v\} \cup \{x \in V : (v, x) \in E\}$. I want to show that either $|V| < 13$ or $|N(u)| < |V| - 6$ (so $|\{w \in V : (u, w) \in E\}| = d(u) < |V| - 7$) or $|N(v)| < |V| - 6$ (so $|\{x \in V : (v, x) \in E\}| = d(v) < |V| - 7$).

body: There are three cases to consider:

case $|V| < 13$: In this case the disjunction we want follows.

case $|V| \geq 13 \wedge N(u) < |V| - 6$: In this case the disjunction we want follows.

case $|V| \geq 13 \wedge N(u) \geq |V| - 6$: Since u and v are disconnected, $N(u)$ and $N(v)$ are disjoint, so the vertices of $N(v)$ must be in $V \setminus N(u)$. The number of vertices available to $N(v)$ is therefore:

$$\begin{aligned} N(v) &\leq |V| - |N(u)| \leq |V| - (|V| - 6) = 6 \\ d(v) &= N(v) - 1 \leq 5 \\ |V| &\geq 13 \\ |V| - 7 &\geq 13 - 7 = 6 > 5 \geq d(v) \end{aligned}$$

In this case the disjunction we want follows.

In all three possible cases the conclusion follows. ■

(e) **[3 marks]** Prove that every undirected graph $G = (V, E)$ with every vertex having degree at least 5, has a cycle.

Solution

sample solution: I will prove:

$$\forall G = (V, E), (\forall v \in V, d(v) \geq 5) \Rightarrow G \text{ has a cycle.}$$

header: Let $G = (V, E)$ be an arbitrary undirected graph. Assume that every vertex in V has degree at least 5. I want to show that there is a cycle in G.

body: Let $v \in V^*$ Let $P = v = v_0, \ldots, v_k$ be a path of maximum length, k starting at v. I know that such a maximum exists because with no more than $|V|$ distinct vertices available for a path, the length can be no more than $|V| - 1$. Also, $k \geq 2$, since v_0 has at least 5 neighbours, and each of these 5 neighbours has at least 4 neighbours other than v_0, so the maximum length is no smaller than 2. Since $d(v_k) \geq 5$ vertex v_k has at least 4 neighbours other than its predecessor v_{k-1}. All of v_k's neighbours must lie on P, since otherwise we could extend the length of P by one, contradicting our choice of a maximal path. Choose $v_i \in P$ so that $i \neq k - 1$ and v_i is a neighbour of v_k. Then $v_k, v_i, \ldots, v_{k-1}, v_k$ is a cycle with edges (v_k, v_i) and $(v_i, v_i + 1), \ldots, (v_{k-1}, v_k)$, and at least 3 distinct vertices since $v_i \neq v_{k-1}$ and $v_i \neq v_k$. ■.

*I am assuming that G is non-empty

(f) **[3 marks]** Prove or disprove: If $G = (V, E)$ is an undirected graph where every vertex has degree at least 4 and $u \in V$, then there are at least 64 distinct paths in G that start at u.

Solution

statement: I will prove:

$$\forall G = (V, E), (\forall v \in V, d(v) \geq 4) \Rightarrow \forall u \in V, \exists S = \{p : p \text{ is a path originating at } u\} w \wedge |S| \geq 64$$

header: Let $G = (V, E)$ be an arbitary undirected graph. Assume that each vertex in G has degree at least 4. Let u be an arbitrary vertex in G, and let $S = \{p : p \text{ is a path originating at } u\}$. I want to show that $|S| \geq 64$.

body: Since $d(u) \geq 4$, u has (at least) 4 neighbours, so there are 4 paths of length 1 in S. Each of the terminal vertices of the 4 paths of length 1 in S also have 4 neighbours, so each have at least 3 neighbours other than u, so there are 12 paths of length 2 in S. Each of the terminal vertices of the 12 paths of length 2 in S have at least 2 neighbours other than their 2 predecessors on their path, so there are 24 paths of length 3 in S. Finally, each of the terminal vertices of the 24 paths of length 3 in S have at least one neighbour other than the 3 that precede them in their path, so there are 24 paths of length 4 in S. Each of these paths is distinct owing to differing in at least one vertex, so $|S| \geq 4 + 12 + 24 + 24 = 64$ paths without even counting the path u itself (of length 0). ■