CSC165H1: Problem Set 3

Due November 13, before 4 p.m.

General instructions

Please read the following instructions carefully before starting the problem set. They contain important information about general problem set expectations, problem set submission instructions, and reminders of course policies.

- Your problem sets are graded on both correctness and clarity of communication. Solutions which are technically correct but poorly written will not receive full marks. Please read over your solutions carefully before submitting them. Proofs should have headers and bodies in the form described in the course note.
- Each problem set may be completed in groups of up to three. If you are working in a group for this problem set, please consult `https://github.com/MarkUsProject/Markus/wiki/Student_Groups` for a brief explanation of how to create a group on MarkUs.

  Exception: Problem Sets 0 and 1 must be completed individually.
- Solutions must be typeset electronically, and submitted as a PDF with the correct filename. **Handwritten submissions will receive a grade of ZERO**.

  The required filename for this problem set is **problem_set3.pdf**.
- Problem sets must be submitted online through MarkUs. If you haven't used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it! If you are working with a partner, you must form a group on MarkUs, and make one submission per group. "I didn't know how to use MarkUs" is not a valid excuse for submitting late work.
- Your submitted file should not be larger than 9MB. This may happen if you are using a word processing software like Microsoft Word; if it does, you should look into PDF compression tools to make your PDF smaller, although please make sure that your PDF is still legible before submitting!
- The work you submit for credit must be your own; you may not refer to or copy from the work of other groups, or external sources like websites or textbooks. You may, however, refer to any text from the Course Notes (or posted lecture notes), except when explicitly asked not to.

Additional instructions

- For each proof, clearly define the predicate ($P(n)$) that is relevant for your proof.
- Some proofs required for this problem set may not require induction.
- You may not use forms of induction we have not covered in lecture, in other words strong induction is off-limits unless otherwise indicated.
- Please follow the same guidelines as Problem Set 2 for all proofs.

1. **[9 marks] build up a result...**

   (a) **[3 marks]** Suppose a, b, c are integers with $ab \neq 0$, $\gcd(|a|, |b|) = 1$, and $a \mid bc$. Prove that $a \mid c$. **Hint**: Use Theorem 2.2 on page 56 of the course notes.

   (b) **[3 marks]** You may assume without proof that if $n \in \mathbb{N}$ is the size of set S, and $k \in \mathbb{N}$ has $0 \leq k \leq n$, the expression

   $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

   called "n choose k" is the number of subsets of S of size k. This is an integer, indeed it is a natural number. Prove that if p is a prime, and $1 < k < p$, then $\binom{p}{k}$ is divisible by p. **Hint**: Use the previous part of this question.

   (c) **[3 marks]** Suppose $n, p \in \mathbb{N}$, and that p is prime. Prove that $n^p \equiv n \pmod{p}$. **Hint**: Try induction on n, and you may use this instance of the Binomial Theorem without proof:

   $$(n+1)^p = \sum_{k=0}^{k=p} \binom{p}{k} n^{p-k} \cdot 1^k$$

2. **[9 marks] big greek letters...**

   (a) **[3 marks]** Suppose $b \in \mathbb{R}^+$. Prove, without using Theorem 5.1 from page 86, that $bn^2 \in \mathcal{O}(2^n)$ but $bn^2 \notin \Omega(2^n)$. **Hint**: You may want to take logarithms and consider a starting point where n is an integer power of 2.

   (b) **[3 marks]** Prove or disprove: For all $f, g : \mathbb{N} \to \mathbb{R}^+$, that $f \notin \Omega(g) \Rightarrow f \in \mathcal{O}(g)$.

   (c) **[3 marks]** Prove or disprove: For all $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{R}^+$, that $f_1 \in \Theta(g_1) \wedge f_2 \in \Theta(g_2) \Rightarrow \max(f_1, f_2) \in \Theta(\max(g_1, g_2))$, where $\max(f_1, f_2)(n) = \max(f_1(n), f_2(n))$.

3. **[3 marks] code...** Devise an exact formula for **repeat(n)** below. You can check your results by running the code in a Python interpreter (version 3.6 or higher). Explain how you derived your formula. Simplify your calculations by stating a big-Θ asymptotic bound for **repeat(n)**, and explain your reasoning.

```python
def repeat(n: int) -> int:
    summation = 0
    for i in range(n):
        for j in range(n**(i % 3)):
            summation = summation + 1
    return summation
```

4. **[9 marks] n to $n+1$...**

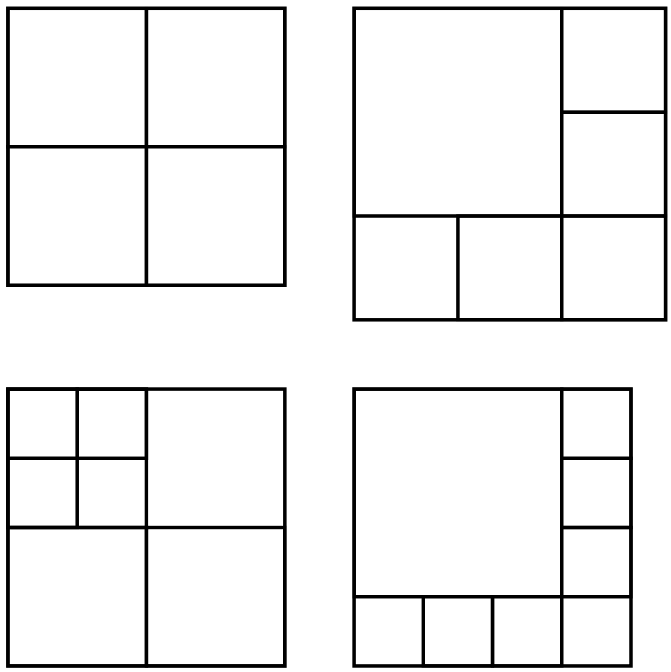
   (a) **[3 marks]** Use induction to prove that for $n \in \mathbb{N}$, 6 divides $5^{2n+1} + 1$. You may not use Ex. 2.19 on modular arithmetic.

   (b) **[3 marks]** Use induction to prove that for any natural number n,

   $$\sum_{i=0}^{i=n} (1/7)^i = \frac{7 - (1/7)^n}{6}$$

   You may not use any formulas for geometric series.

(c) **[3 marks]** Notice below that any square could be tiled (exactly covered without overlapping) using four squares that are half its width. It could also be tiled using six smaller squares: one that is 2/3 its width tucked into the upper-left corner, and 5 that are 1/3 its width across the bottom and up the right margin. Or it could be tiled using seven smaller squares: three that are half its width, plus four that are 1/4 its width. Or it could be tiled using eight smaller squares: one that is 3/4 its width tucked in the top-left corner, plus seven that are 1/4 its width bordering the larger one on the bottom and right.



Assume the facts in the above paragraph, or make some drawings if you are unconvinced. Then use those assumptions, and induction on n, to prove the following three facts:

Define $P_6(n)$: "Any square can be tiled by $6 + 3n$ smaller squares." Prove $\forall n \in \mathbb{N}, P_6(n)$.

Define $P_7(n)$: "Any square can be tiled by $7 + 3n$ smaller squares." Prove $\forall n \in \mathbb{N}, P_7(n)$.

Define $P_8(n)$: "Any square can be tiled by $8 + 3n$ smaller squares." Prove $\forall n \in \mathbb{N}, P_8(n)$.

Finally, use all three facts to prove that any square can be tiled by m smaller squares, provided $m \geq 6$ is an integer.