CSC384
Knowledge Representation
Part 1

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Credits

These slides are drawn from or inspired by a multitude of sources including :

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Faheim Bacchus
Michael Winter
Hector Levesque

Introduction

What is Knowledge Representation and Reasoning (KR&R)?
Symbolic encoding of propositions believed by some agent and their manipulation to produce propositions that are believed by the agent but not explicitly stated.

Why KR&R:

- Large amounts of knowledge are used to understand the world around us.
- **Reasoning** provides compression in the knowledge we need to store.
- **Without** reasoning we would have to store an infeasible amount of information:
  Example: Elephants can’t fit into teacups, Elephants can’t fit into cars, instead of just knowing that larger objects can’t fit into smaller objects.

Introduction

- **Information:**
  (1) Block A is above block B;
  (2) Block B is above block C.

- **Query:** Is A above C?

Given the information, human can easily draw the conclusion.
How can a **machine** do the same?

Introduction

- Tony, Mike, and John are members of the Alpine Club.
- Every member of the Alpine Club who is not a skier is a mountain climber.
- Mountain climbers do not like rain, and anyone who does not like snow is not a skier.
- Mike dislikes whatever Tony likes, and likes whatever Tony dislikes.
- Tony likes rain and snow.
- Is there a member of the Alpine Club who is a mountain climber but not a skier?

Logical Representations for KR

Logical representations

- are mathematically precise; thus it's possible to analyze their limitations, properties, and complexity of inferences.
- are formal languages; thus computer programs can manipulate sentences in the language.
- typically, have well-developed proof theories: formal procedures for reasoning to produce new sentences.

In this module we will study **First-Order logic (FOL)**, and a reasoning mechanism called **resolution** that operates on First-Order logic.

Review: Propositional Logic – Syntax

Propositional Variable: A variable which takes only **True** or **False** as values.

The set of all propositional formulas is defined recursively as follows:

- Every propositional variable is a propositional formula;
- If φ is a propositional formula, then so is $\neg\varphi$;
- If φ_1 and φ_2 are propositional formulas, then so are
  - $\varphi_1 \wedge \varphi_2$ (**Conjunction**);
  - $\varphi_1 \vee \varphi_2$ (**Disjunction**);
  - $\varphi_1 \rightarrow \varphi_2$ (**Implication**);
  - $\varphi_1 \leftrightarrow \varphi_2$ (**Bi-implication**).

Review: Propositional Logic – Semantic

Truth Assignment: A function τ from the propositional variables into the set of truth values $\{T, F\}$.

Let τ be a truth assignment. The extension $\bar{\tau}$ of τ assigns either T or F to every formula and is defined as follows:

- If $A = x$, where x is a variable, then $\bar{\tau}(A) = \tau(x)$.
- $\bar{\tau}(\neg A) = T$ iff $\bar{\tau}(A) = F$;
- $\bar{\tau}(A \wedge B) = T$ iff $\bar{\tau}(A) = T$ and $\bar{\tau}(B) = T$;
- $\bar{\tau}(A \vee B) = T$ iff $\bar{\tau}(A) = T$ or $\bar{\tau}(B) = T$;
- $\bar{\tau}(A \rightarrow B) = F$ iff $\bar{\tau}(A) = T$ and $\bar{\tau}(B) = F$.

Review: Propositional Logic – Semantic

Example: Let $V = \{p, r, q\}$ be a set of propositional variables and $\tau_1 : V \to \{T, F\}$ and $\tau_2 : V \to \{T, F\}$ be two truth assignments s.t.:

- $\tau_1(p) = T$, $\tau_1(q) = F$, $\tau_1(r) = F$.
- $\tau_2(p) = F$, $\tau_2(q) = T$, $\tau_2(r) = F$.

Then

$$\bar{\tau}_1((\underbrace{\neg p}_{F} \land \underbrace{q}_{T}) \to r) = T$$

$$\bar{\tau}_2(\overbrace{(\underbrace{\neg p}_{T} \land \underbrace{q}_{T})}^{T} \to \underset{F}{\underset{\downarrow}{r}}) = F$$

Review: Propositional Logic – Semantic

A truth assignment τ **satisfies** a formula A iff $\bar{\tau}(A) = T$.
τ satisfies a set Φ of formulas iff τ satisfies all formula in Φ.

A set Φ of formulas is satisfiable iff some truth assignment τ satisfies Φ.
Otherwise, Φ is unsatisfiable.

Example:

$\Phi_1 = \{\underset{F}{r} \to \underset{F}{(p \wedge q)}, \underset{T}{\neg p}\}$

$\tau(r) = F$ $\tau(P) = F$ $\tau(q) = T$

$\Phi_2 = \{\underset{T}{r} \to \underset{F}{\overset{F}{\overbrace{(p \wedge q)}}}, \underset{T}{r} \wedge \underset{T}{\underbrace{\neg p}}\}$

unsatisfiable

Review: Propositional Logic – Semantic

A formula A is a **logical consequence** of Φ (denoted by $\Phi \models A$) iff for every truth assignment τ, if τ satisfies Φ, then τ satisfies A.

Example: Let $\Phi = \{\underset{\overline{T}}{r} \rightarrow (\overbrace{(\underset{\overline{T}}{p} \wedge q)}^{T} \overset{\text{or}}{\vee} \overbrace{s}^{T}), \underset{\overline{T}}{r} \wedge \underset{\overline{T}}{p}\}$.

$p \wedge q = T \rightarrow q = T$

or

$s = T$

Then $\Phi \models q \vee s$

Limitations of Propositional Language

- **Only Boolean variables**: Without non-Boolean variables **cross references between individuals** in statements are **impossible**.
  Example: 'If a person has a sibling and that sibling has a child, then the person is an aunt or an uncle.'
  S: a person has a sibling.
  C: a sibling has a child.
  A: a person is an aunt or an uncle.

  $S \wedge C \rightarrow A$

  This approach doesn't work:
  person in S and A are not related.
  sibling in S and C are not related.

- **No quantifiers:** To state a property for all (or some) members of the domain we have to **explicitly list** them.
  Example: 'Every member of the Alpine Club who is not a skier is a mountain climber'

First-Order Logic: Syntax

For **first-order logic** following components are required:

- A set V of variables.
- A set F of function symbols.
- A set P of predicate (relation) symbols.

- **Functions** and **variables** are used to construct terms.
- **Predicates** are defined over terms.
- **Predicates** and **terms** are used to construct formulas.

A set $\mathcal{L}$ of **function** and **predicate symbols** is called a first-order vocabulary.

First-Order Logic: Intuition

- Terms (variables and functions) denote elements of the domain.
- Atomic formulas denote properties and relations that hold about the elements in the domain.
- Other formulas generate more complex assertions by composing atomic formulas.

First-Order Logic: Syntax

Let $\mathcal{L}$ be a set of function and predicate symbols.

1. Every variable is a term.
2. If f is an n-ary function symbol in $\mathcal{L}$ and $t_1, t_2, ..., t_n$ are $\mathcal{L}$-terms, then $f(t_1, t_2, ..., t_n)$ is a $\mathcal{L}$-term.

Note: 0-ary functions symbols are called **constant symbols**.

Example:

$f(x, g(x, y))$

$f(c_1, c_2)$, c_1 and c_2 are constants

First-Order Logic: Syntax

Let $\mathcal{L}$ be a vocabulary. The set of first-order $\mathcal{L}$-formulas is defined recursively:

1. Atomic Formula: $P(t_1, t_2, ..., t_n)$, where P is an n-ary predicate symbol in $\mathcal{L}$ and $t_1, t_2, ..., t_n$ are $\mathcal{L}$-terms.

2. Negation: $\neg f$, where f is a $\mathcal{L}$-formula.

3. Conjunction: $f_1 \wedge f_2 \wedge ... \wedge f_n$, where $f_1, f_2, ..., f_n$ are $\mathcal{L}$-formulas.

4. Disjunction: $f_1 \vee f_2 \vee ... \vee f_n$, where $f_1, f_2, ..., f_n$ are $\mathcal{L}$-formulas.

5. Implication: $f_1 \rightarrow f_2$, where f_1, f_2 are $\mathcal{L}$-formulas.

6. Existential: $\exists x f$, where x is a variable and f is a $\mathcal{L}$-formula.

7. Universal: $\forall x f$, where x is a variable and f is a $\mathcal{L}$-formula.

Converting English to First-Order Language

- **Individuals**: Constants (0-ary Functions)
  - **tony, mike, john**
    rain, snow
- **Types**: Unary Predicates
  - $AC(x)$: x belongs to Alpine Club.
  - $S(x)$: x is a skier.
  - $C(x)$: x is a mountain climber.
- **Relationships**: Binary Predicates
  - $L(x, y)$: x likes y.

Converting English to First-Order Language

- **Basic Facts**:
  - Tony, Mike, and John belong to the Alpine Club:
    $AC(\textbf{tony}), AC(\textbf{mike}), AC(\textbf{john})$
  - Tony likes rain and snow:
    $L(\textbf{tony, rain}), L(\textbf{tony, snow})$
- **Complex Facts**:
  - Every member of the Alpine Club who is not a skier is a mountain climber.

$$\forall x \, [\, AC(x) \wedge \neg S(x) \rightarrow C(x) \,]$$

  - Mountain climbers do not like rain, and anyone who does not like snow is not a skier.

$$\forall x \, [\, C(x) \rightarrow \neg L(x, rain) \,] \wedge \forall x \, [\, \neg L(x, snow) \rightarrow \neg S(x) \,]$$

Converting English to First-Order Language

- Mike dislikes whatever Tony likes, and likes whatever Tony dislikes.

$$\forall x[L(tony,x) \rightarrow \neg L(mike,x)] \land \forall x[\neg L(tony,x) \rightarrow L(mike,x)]$$

- Is there a member of the Alpine Club who is a mountain climber but not a skier?

$$\exists x[AC(x) \land C(x) \land \neg S(x)]$$

First-Order Logic: Syntax

Like variables in programming languages, the variables in FOL have a scope which is determined by the quantifiers.

Lexical scope for variables:

$$Animal(x) \land \underline{\exists x}\underbrace{[Human(x) \lor Women(x)]\ .}$$

$$\underline{\exists x}\underbrace{[Animal(x) \rightarrow \neg Human(x)]} \land \underline{\exists x}\underbrace{[Human(x) \lor Women(x)]}$$

First-Order Logic: Semantic

- In the **propositional logic**, a **truth assignment** provides meaning to a formula.
- In **FOL** we can talk about **(non-Boolean) individuals and elements**.
  So the simple universe of truth values is not rich enough to provide a suitable interpretation for FOL formulas.
- We need more **more complicated objects** to give meaning to formulas and terms.
- These objects are called **structures**.

First-Order Structures

Let $\mathcal{L}$ be a first-order vocabulary. An $\mathcal{L}$-**structure** $\mathcal{M}$ consists of the following:

1. A **nonempty set** M called the universe (domain) of discourse.

2. For each n-ary **function symbol** $f \in \mathcal{L}$, an associated function $f^{\mathcal{M}} : M^n \to M$.
   Note: If $n = 0$, then f is a constant symbol and $f^{\mathcal{M}}$ is simply an element of M.
   $f^{\mathcal{M}}$ is called the **extension** of the function symbol f in $\mathcal{M}$.

3. For each n-ary **predicate symbol** $P \in \mathcal{L}$, an associated relation $P^{\mathcal{M}} \subseteq \mathcal{M}^n$.
   $P^{\mathcal{M}}$ is called the **extension** of the predicate symbol P in $\mathcal{M}$.

First-Order Structures: Example

Blocks World:

Suppose $\mathcal{L}_{BW}$ includes the following symbols:

- **Function Symbols:**
  - $under(x)$: the block immediately under x if x is not on table; x itself otherwise.
- **Predicate Symbols:**
  - $on(x, y)$: x is place (directly) on y.
  - $above(x, y)$: x is above y.
  - $clear(x)$: no blocks are above x.
  - $ontable(x)$: no blocks are under x.

Suppose $\mathcal{L}_{BW}$ includes the following symbols:

- **Function Symbols:**
  - $under(x)$: the block immediately under x if x is not on table; x itself otherwise.
- **Predicate Symbols:**
  - $on(x, y)$: x is place (directly) on y.
  - $above(x, y)$: x is above y.
  - $clear(x)$: no blocks are above x.
  - $ontable(x)$: no blocks are under x.

$\mathcal{M}_1$ is a $\mathcal{L}_{BW}$-structure such that:

$M_1 = \{A, B, C, D\}$

$on^{\mathcal{M}_1} = \{\langle A, B\rangle, \langle B, C\rangle\}$

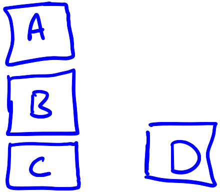
$above^{\mathcal{M}_1} = \{\langle A, B\rangle, \langle B, C\rangle, \langle A, C\rangle\}$

$clear^{\mathcal{M}_1} = \{A, D\}$

$ontable^{\mathcal{M}_1} = \{C, D\}$

$under^{\mathcal{M}_1}(A) = B$, $under^{\mathcal{M}_1}(B) = C$,

$under^{\mathcal{M}_1}(C) = C$, $under^{\mathcal{M}_1}(D) = D$



Suppose $\mathcal{L}_{BW}$ includes the following symbols:

- **Function Symbols:**
  - $under(x)$: the block immediately under x if x is not on table; x itself otherwise.
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  - $on(x, y)$: x is place (directly) on y.
  - $above(x, y)$: x is above y.
  - $clear(x)$: no blocks are above x.
  - $ontable(x)$: x is placed on the table.

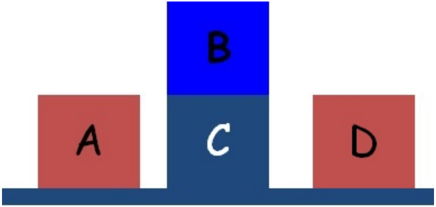
$\mathcal{M}_2$:

$M_2 = \{A, B, C, D\}$

$on^{\mathcal{M}_2} = \{\langle B, C\rangle\}$

$above^{\mathcal{M}_2} = \{\langle B, C\rangle\}$

Represent the following configuration by a $\mathcal{L}_{BW}$-structure.



$Clear^{\mathcal{M}_2} = \{A, B, D\}$

$ontable^{\mathcal{M}_2} = \{A, C, D\}$

$\text{under}^{M_2}(A) = A$ $\text{under}^{M_2}(B) = C$

$\text{under}^{M_2}(C) = C$ $\text{under}^{M_2}(D) = D$

Semantic of First-Order Logic: Intuition

Every $\mathcal{L}$-formula becomes either true or false when interpreted by an $\mathcal{L}$-structure $\mathcal{M}$.

That is, the truth value of a first-order formulas A is evaluated w.r.t to a first-order structure $\mathcal{M}$:

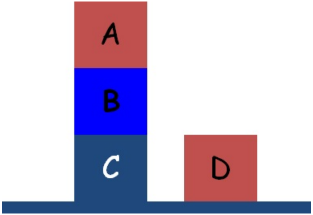
- Terms (variables and functions) of a formula denote elements of the domain.
  So every term in A must correspond with an element of the universe of $\mathcal{M}$.

- Atomic formulas denote properties and relations that hold about the elements in the domain.
  $P(t_1, ..., t_n)$ is true in $\mathcal{M}$ if $t_1, ..., t_n$ are related to each other by $P^{\mathcal{M}}$.

- Other formulas generate more complex assertions by composing atomic formulas.
  Their truth is dependent on the truth of the atomic formulas in them.

Semantic of First-Order Logic: Variable Assignments

Let $\mathcal{M}$ be a structure and X be a set of variables. An **object assignment** σ for $\mathcal{M}$ is a **mapping** from variables in X to the universe of $\mathcal{M}$.



$$X = \{v_1, v_2, v_3, v_4\}$$

$$\sigma(v_1) = D, \qquad \sigma(v_2) = C$$
$$\sigma(v_3) = B, \qquad \sigma(v_4) = A$$

Semantic of First-Order Logic: Variable Assignments

Remember the recursive definition of term:
Let $\mathcal{L}$ be a set of function and predicate symbols.

1. Every variable x is a term.
2. If f is an n-ary function symbol in $\mathcal{L}$ and $t_1, t_2, ..., t_n$ are $\mathcal{L}$-terms, then $f(t_1, t_2, ..., t_n)$ is a $\mathcal{L}$-term.

Let $\mathcal{L}$ be a vocabulary and $\mathcal{M}$ be an $\mathcal{L}$-structure.
The extension $\bar{\sigma}$ of σ is defined recursively:

1. for every variable x, $\bar{\sigma}(x) = \sigma(x)$;
2. for every function symbol $f \in \mathcal{L}$, $\bar{\sigma}(f(t_1, ..., t_n)) = f^{\mathcal{M}}(\bar{\sigma}(t_1), ..., \bar{\sigma}(t_n))$.

Semantic of First-Order Logic: Variable Assignments

Let $\mathcal{L}$ be a vocabulary and $\mathcal{M}$ be an $\mathcal{L}$-structure.
The extension $\bar{\sigma}$ of σ is defined recursively:

1. for every variable x, $\bar{\sigma}(x) = \sigma(x)$;
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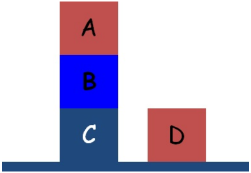

$$under^{\mathcal{M}}(A) = B \qquad under^{\mathcal{M}}(B) = C$$
$$under^{\mathcal{M}}(C) = C \qquad under^{\mathcal{M}}(D) = D$$

$X = \{v_1, v_2, v_3, v_4\}$
$\sigma(v_1) = D, \qquad \sigma(v_2) = C$
$\sigma(v_3) = B, \qquad \underline{\sigma(v_4) = A} \rightarrow D$

$$\bar{\sigma}(under(under(v_4))) = under^{\mathcal{M}}(\underbrace{\bar{\sigma}(under(v_4))}_{B}) = under^{\mathcal{M}}(\overset{D}{B}) = \overset{D}{C}$$

$$\overline{\sigma}\,(under(v_4)) = under^{M}(\underbrace{\overline{\sigma}\,(v_4)}_{A}) = under^{M}(\underset{\underset{D}{\downarrow}}{A}) = \underset{\underset{D}{\downarrow}}{B}$$

$$\overline{\sigma}\,(v_4) = \sigma(v_4) = \underset{\underset{D}{\downarrow}}{A}$$

First-Order Logic Semantic: Models (Interpretations)

For an $\mathcal{L}$-formula A, $\mathcal{M} \models A[\sigma]$ ($\mathcal{M}$ **satisfies** A under σ, or $\mathcal{M}$ is a **model** of A under σ) is defined recursively on the structure of A as follows:

$$\begin{array}{lll}
\mathcal{M} \models P(t_1, ..., t_n)[\sigma] & \text{iff} & \langle \bar{\sigma}(t_1), ..., \bar{\sigma}(t_n) \rangle \in P^{\mathcal{M}}. \\
\mathcal{M} \models (s = t)[\sigma] & \text{iff} & \bar{\sigma}(s) = \bar{\sigma}(t). \\
\mathcal{M} \models \neg A[\sigma] & \text{iff} & \mathcal{M} \not\models A[\sigma]. \\
\mathcal{M} \models (A \vee B)[\sigma] & \text{iff} & \mathcal{M} \models A[\sigma] \text{ or } \mathcal{M} \models B[\sigma]. \\
\mathcal{M} \models (A \wedge B)[\sigma] & \text{iff} & \mathcal{M} \models A[\sigma] \text{ and } \mathcal{M} \models B[\sigma]. \\
\mathcal{M} \models (\forall x A)[\sigma] & \text{iff} & \mathcal{M} \models A[\sigma(m/x)] \text{ for all } m \in M. \\
\mathcal{M} \models (\exists x A)[\sigma] & \text{iff} & \mathcal{M} \models A[\sigma(m/x)] \text{ for some } m \in M.
\end{array}$$

First-Order Logic Semantic: Models (Interpretations)

For an $\mathcal{L}$-formula A, $\mathcal{M} \models A[\sigma]$ ($\mathcal{M}$ **satisfies** A under σ, or $\mathcal{M}$ is a **model** of A under σ) is defined recursively on the structure of A as follows:

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\mathcal{M} \models (\forall x A)[\sigma] & \text{iff} & \mathcal{M} \models A[\sigma(m/x)] \text{ for all } m \in M. \\
\mathcal{M} \models (\exists x A)[\sigma] & \text{iff} & \mathcal{M} \models A[\sigma(m/x)] \text{ for some } m \in M.
\end{array}
$$

Note: $\sigma(m/x)$ is a object variable assignment function. Exactly like σ, but maps the variable x to the individual $m \in M$. That is:

For $y \neq x : \sigma(m/x)(y) = \sigma(y)$

For x: $\sigma(m/x)(x) = \sigma(m)$

Models: Example

Let $\mathcal{M}_3$ be a structure such that:
$M_3 = \{A, B, C, D\}$
$on^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle\}$
$above^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle, \langle A, C\rangle\}$
$clear^{\mathcal{M}_3} = \{A, D\}$
$ontable^{\mathcal{M}_3} = \{C, D\}$

Does $\mathcal{M}_3$ satisfy
$\forall x \forall y (on(x, y) \rightarrow above(x, y))$ ✓

$x = A$
$y = A$ ✓
$y = B$ ✓
$y = C$ ✓
$y = D$ ✓

$x = B$
$y = A$ ✓
$y = B$ ✓
$y = C$ ✓
$y = D$ ✓

$x = C$
✓ $y = A$
$y = B$
✓ $y = C$
✓ $y = D$

$x = D$
✓ $y = A$
✓ $y = B$
✓ $y = C$
✓ $y = D$

Let $\mathcal{M}_3$ be a structure such that:
$M_3 = \{A, B, C, D\}$
$on^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle\}$
$above^{\mathcal{M}_3} = \{\underline{\langle A, B\rangle,} \langle B, C\rangle, \langle A, C\rangle\}$
$clear^{\mathcal{M}_3} = \{A, D\}$
$ontable^{\mathcal{M}_3} = \{C, D\}$

Does $\mathcal{M}_3$ satisfy
$\forall x \forall y(above(x, y) \rightarrow on(x, y))$ ✗

$x = A$ $y = C$ ✗

$\langle A, C\rangle \in above^{M_3}$

$\langle A, C\rangle \in On^{M_3}$

Let $\mathcal{M}_3$ be a structure such that:

$M_3 = \{A, B, C, D\}$

$on^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle\}$

$above^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle, \langle A, C\rangle\}$

$clear^{\mathcal{M}_3} = \{A, D\}$

$ontable^{\mathcal{M}_3} = \{C, D\}$

Does $\mathcal{M}_3$ satisfy

$\forall x \exists y (clear(x) \vee On(y, x))$ ✓

$x = A$ $y = A$ ✓

$x = B$ $y = A$ ✓

$x = C$ $y = B$ ✓

$x = D$ $y = A$ ✓

Let $\mathcal{M}_3$ be a structure such that:

$M_3 = \{A, B, C, D\}$

$on^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle\}$

$above^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle, \langle A, C\rangle\}$

$clear^{\mathcal{M}_3} = \{A, D\}$

$ontable^{\mathcal{M}_3} = \{C, D\}$

Does $\mathcal{M}_3$ satisfy

$\exists y \forall x (clear(x) \vee On(y, x))$ X

$y = A$ $x = C$ X

$y = B$ $x = B$ X

$y = C$ $x = C$ X

$y = D$ $x = D$ X

First-Order Logic Semantic: Models

An occurrence of x in A is **bound** iff it is in a sub-formula of A of the form $\forall x B$ or $\exists x B$. Otherwise the occurrence is **free**.

Example:

$P(x) \wedge \exists x[P(x) \vee Q(x)]$

free   bounded

In a structure $\mathcal{M}$, formulas with **free variables** might be **true for some** object assignments to the free variables and **false for others**.

Example: Consider the formula $\overbrace{P(x, y) \wedge P(y, x)}^{A}$ and the following structure $\mathcal{M}$:

$M = \{a, b\} \qquad P^{\mathcal{M}} = \{\langle a, a\rangle\}$

$\sigma_1(x) = a \qquad \sigma_1(y) = a \qquad \mathcal{M} \models A[\sigma_1]$

$\sigma_2(x) = a \qquad \sigma_2(y) = b \qquad \mathcal{M} \not\models A[\sigma_2]$

First-Order Logic Semantic: Models

A formula A is **closed** if it contains no free occurrence of a variable.
A **closed formula** is called a **sentence**.
Example:
$P(x) \wedge \exists x[P(x) \vee Q(x)]$.

$\forall x P(x) \wedge \exists x[P(x) \vee Q(x)]$

If σ and σ' agree on the **free variables** of A, then $\mathcal{M} \models A[\sigma]$ iff $\mathcal{M} \models A[\sigma']$.
Proof: Structural induction on A.

Corollary: If A is a **sentence**, then for any object assignments σ and σ',

$$\mathcal{M} \models A[\sigma] \qquad \text{iff} \qquad \mathcal{M} \models A[\sigma']$$

So, if A is a **sentence** (no free variables), σ is **irrelevant** and we omit mention of σ and simply write $\mathcal{M} \models A$.

Logical Satisfiability

Let Φ be a **set of sentences**.

- $\mathcal{M}$ satisfies Φ (denoted by $\mathcal{M} \models \Phi$) if for **every** sentence $A \in \Phi$, $\mathcal{M} \models A$.
- If $\mathcal{M} \models \Phi$, we say $\mathcal{M}$ is a model of Φ.
- We say that Φ is satisfiable if there is a structure $\mathcal{M}$ such that $\mathcal{M} \models \Phi$.

Models of Logical Sentences: Example

Let Φ_1 be a set containing the following sentences ($\boldsymbol{c_1}, \boldsymbol{c_2}$ are constant symbols, we use **bold** font to distinguish constant symbols from variables):

- $on(\boldsymbol{c_1}, \boldsymbol{c_2})$
- $clear(\boldsymbol{c_1})$
- $above(\boldsymbol{c_1}, \boldsymbol{c_2})$

Construct **two models** of Φ_1 with **size three** (i.e., the size of the domain of each model must be three).

$M_1 = \{A, B, C\}$

$\boldsymbol{c_1}^{\mathcal{M}_1} = A \qquad \boldsymbol{c_2}^{\mathcal{M}_1} = B$

$on^{\mathcal{M}_1} = \{\langle A, B\rangle, \langle B, C\rangle\}$

$clear^{\mathcal{M}_1} = \{A, C\}$

$above^{\mathcal{M}_1} = \{\langle A, B\rangle\}$

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Models of Logical Sentences: Practice Question

Let Φ_2 be a set containing the following sentences($\boldsymbol{c_1}, \boldsymbol{c_2}$ are constant symbols):

- $\forall x(clear(x) \rightarrow \neg\exists y(on(y,x)))$
- $\forall x \forall y(on(x,y) \rightarrow above(x,y))$
- $\forall x \forall y \forall z((above(x,y) \wedge above(y,z)) \rightarrow above(x,z))$
- $on(\boldsymbol{c_1}, \boldsymbol{c_2})$
- $clear(\boldsymbol{c_1})$
- $above(\boldsymbol{c_1}, \boldsymbol{c_2})$

Construct **two models** of Φ_2 with **size three** (i.e., the size of the domain of each model must be three).

$M_2 = \{A, B, C\}$ $\quad c_1^{M_2} = A$ $\quad C_2^{M_2} = B$

$On^{M_2} = \{\langle A, B\rangle, \langle B, C\rangle\}$

$clear^{M_2} = \{A\}$

$above^{M_2} = \{\langle A, B\rangle, \langle B, C\rangle, \langle A, C\rangle\}$

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Logical Satisfiability: Practice Question

Example: is $\{\forall x(P(x) \rightarrow Q(x)), P(\textbf{a}), \neg Q(\textbf{a})\}$ satisfiable?