CSC384
Constraint Satisfaction Problems
Part 1

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Credits

CSP slides are drawn from or inspired by a multitude of sources including :

Faheim Bacchus
Sheila McIlraith
Andrew Moore
Hojjat Ghaderi
Craig Boutillier
Jurgen Strum
Shaul Markovitch

Constraint Satisfaction Problems (CSPs)

- Chapter 6
  - 6.1: Formalism
  - 6.2: Constraint Propagation
  - 6.3: Backtracking Search for CSP
  - 6.4 is about local search which is a very useful idea but we won't cover it in class.

Constraint Satisfaction Problems (CSPs) – Introduction

- **Uninformed search problems**
  - use problem-specific state representations and heuristics;
  - are generally concerned about determining paths from the current state to goal states;
  - view states as black boxes with no internal structures.
- **Constraint Satisfaction Problems (CSPs)**
  - care less about paths and more about final (goal) configurations;
  - take advantage of a general state representation.
  - the uniform state representation allows design of more efficient algorithms.
- Techniques for solving CSPs have many practical applications in industry.

Constraint Satisfaction Problems (CSPs) – Intuition

- Represent **states** as vectors of feature values.[1]
  - A set of k variables (known as **features**).
  - **Each variable** has a domain of different values.
  - A **state** is specified by an assignment of values to all variables.
  - A **partial state** is specified by an assignment of a value to some of the variables.
- A **goal** is specified as conditions on the vector of feature values.
- **Solving a CSP**: find a set of values for the features (variables) so that the values **satisfy** the specified conditions (constraints).

W_1, W_2, W_3

$Dom[W_1] = Dom[W_2] = Dom[W_3] = \{10, 15, 20\}$

$W_1 > W_2$ $\quad W_2 = W_3$

$W_1 > W_3$

[1] Feature vectors provide a general state representation that is useful in many other areas of AI, particularly Machine Learning, Reasoning under Uncertainty, and Computer Vision.

Example: Sudoku

	2							
			6					3
	7	4		8				
					3			2
	8			4			1	
6			5					
				1		7	8	
5					9			
							4	

1	2	6	4	3	7	9	5	8
8	9	5	6	2	1	4	7	3
3	7	4	9	8	5	1	2	6
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

Example: Sudoku

- Each **variable** represent a cell.
- **Domain**: a single value for cells already filled in; the set $\{1, ..., 9\}$ for empty cells.
- **State**: any completed board given by specifying the value in each cell.
- **Partial State**: some incomplete filling out of the board.
- **Constrains**: The variables that form
  - a column must be distinct;
  - a row must be distinct;
  - a sub-square must be distinct.

Formalization of a CSP

A **CSP** consists of

- A set of **variables** $V_1, ..., V_n$;
- A (finite) **domain** of possible values $Dom[V_i]$ for each variable V_i;
- A set of **constraints** $C_1, ..., C_m$.

- Each variable V_i can be assigned any value from its domain:

$$V_i = d \qquad \text{where} \qquad d \in Dom[V_i]$$

- Each constraint C
  - Has a set of variables it operates over, called its **scope**.
    Example: The scope of $C(V_1, V_2, V_4)$ is $\{V_1, V_2, V_4\}$
  - Given an assignment to variables the C returns
    True if the assignment satisfies the constraint;
    False if the assignment falsifies the constraint.

Formalization of a CSP

- **Solution** to a CSP: An assignment of a value to all of the variables such that every constraint is satisfied.

- A CSP is **unsatisfiable** if no solution exists.

- **Unary** Constraints (over one variable)
  $C(X) : X = 2;$
  $C(Y) : Y > 5$

- **Binary** Constraints (over two variables)
  $C(X, Y) : X + Y < 6$

- **Higher-order** constraints: over 3 or more variables.
  $ALL - Diff(V_1, .., V_n)$: $V_1 \neq V_2, V_1 \neq V_3, ..., V_2 \neq V_1, ..., V_n \neq V_1, ..., V_n \neq V_{n-1}$.[2]

[2] Later, we will see that this collection of binary constraints has less pruning power than $ALL - Diff$, so $ALL - Diff$ appears in many CSP problems.

Constraint Table

- We can specify the constraints with a table

$C(1,1,1) = \text{False}$

V1	V2	V4	C(V1,V2,V4)
1	1	1	$\boxed{\text{False}}$
1	1	2	False
1	2	1	False
1	2	2	False
2	1	1	True
2	1	2	False
2	2	1	False
2	2	2	False
3	1	1	False
3	1	2	True
3	2	1	True
3	2	2	False

$C(2,1,1) = \text{True}$

- Often we can specify the constraint more compactly with an expression.

$C(V_1, V_2, V_4) : (V_1 = V_2 + V_4)$

1	2	6	4	3	7	9	5	8
8	9	5	**6**	2	1	4	7	**3**
3	**7**	**4**	9	**8**	5	1	2	6
4	5	7	1	9	**3**	8	6	**2**
9	**8**	3	2	**4**	6	5	**1**	7
6	1	2	**5**	7	8	3	9	4
2	6	9	3	**1**	4	**7**	**8**	5
5	4	8	7	6	**9**	2	3	1
7	3	1	8	5	2	6	**4**	9

- **Variables**: $V_{11}, V_{12}, ..., V_{21}, V_{22}, ..., V_{91}, ..., V_{99}$

- **Domains**: $Dom[V_{ij}] = \{1, 2, .., 9\}$ for empty cells
  $Dom[V_{ij}] = \{k\}$, where k is a fixed value, for filled cells.

- **Constraints**:

  - Row constraints:

$All - Diff(V_{11}, V_{12}, V_{13}, ..., V_{19})$

$All - Diff(V_{21}, V_{22}, V_{23}, ..., V_{29})$

...

$All - Diff(V_{91}, V_{12}, V_{13}, ..., V_{99})$

1	2	6	4	3	7	9	5	8
8	9	5	6	2	1	4	7	3
3	7	4	9	8	5	1	2	6
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

Example: Sudoku

- **Constraints**:

  - Row constraints:

    $All - Diff(V_{11}, V_{12}, V_{13}, ..., V_{19})$

    $All - Diff(V_{21}, V_{22}, V_{23}, ..., V_{29})$

    ...

    $All - Diff(V_{91}, V_{12}, V_{13}, ..., V_{99})$

  - Column Constraints:

    $All - Diff(V_{11}, V_{21}, V_{31}, ..., V_{91})$

    $All - Diff(V_{12}, V_{22}, V_{32}, ..., V_{92})$

    ...

    $All - Diff(V_{19}, V_{29}, V_{39}, ..., V_{99})$

1	2	6	4	3	7	9	5	8
8	9	5	6	2	1	4	7	3
3	7	4	9	8	5	1	2	6
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

Example: Sudoku

- **Constraints**:

  - Row constraints:

$$All - Diff(V_{11}, V_{12}, V_{13}, ..., V_{19})$$
$$All - Diff(V_{21}, V_{22}, V_{23}, ..., V_{29})$$
$$...$$
$$All - Diff(V_{91}, V_{12}, V_{13}, ..., V_{99})$$

  - Column Constraints:

$$All - Diff(V_{11}, V_{21}, V_{31}, ..., V_{91})$$
$$All - Diff(V_{12}, V_{22}, V_{32}, ..., V_{92})$$
$$...$$
$$All - Diff(V_{19}, V_{29}, V_{39}, ..., V_{99})$$

  - Sub-Square Constraints:

$$All{-}Diff(V_{11}, V_{12}, V_{13}, V_{21}, V_{22}, V_{23}, V_{31}, V_{32}, V_{33}),$$
$$...,$$
$$All{-}Diff(V_{77}, V_{78}, V_{79}, ..., V_{97}, V_{98}, V_{99})$$

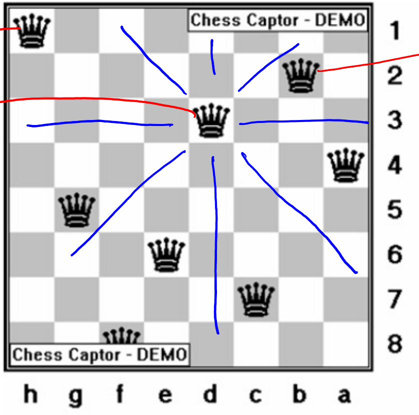
1	2	6	4	3	7	9	5	8
8	9	5	6	2	1	4	7	3
3	7	4	9	8	5	1	2	6
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

Example: N-Queens

Problem Statement: Place N Queens on an $N \times N$ chess board so that no Queen can attack any other Queen.

$Q_1 = 1$

$Q_3 = 21$



$Q_2 = 15$

Example: N-Queens

Problem formulation:

- **Variables:** N variables, each representing a queen
- **Domains:** N^2 values for each variable, representing the of a queen on the chessboard

Number of Possible Configurations: $(N^2)^N$

$\underbrace{N^2} \times \underbrace{N^2} \times \underbrace{N^2} \times \ldots \times \underbrace{N^2}$

For 8-queens:
$(64)^8 = 281,474,976,710,656$
Possible configurations

Example: N-Queens

Is there a better way to represent the N-queens problem? We know we cannot place two queens in a single row.

Problem Statement: Place N Queens on an $N \times N$ chess board so that no Queen can attack any other Queen.

Better Formulation:

- **Variables:** N variables, one for each queen on each row
  Q_i: i-th queen on row i
- **Domains:** Value of Q_i is the column the queen on row i is placed.
  Possible values: $\{1, 2, \dots, N\}$

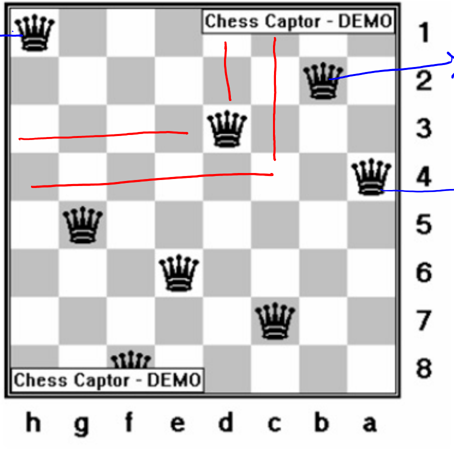
Number of all possible configurations: N^N

$\underbrace{N}\times\underbrace{N}\times\underbrace{N}\ \dots \times\underbrace{N}$

For 8-queens: $8^8 = 16{,}777{,}216$ configurations

Example: N-Queens

$Q_1 = 1$



$Q_2 = 7$

$Q_4 = 8$

Constraints:

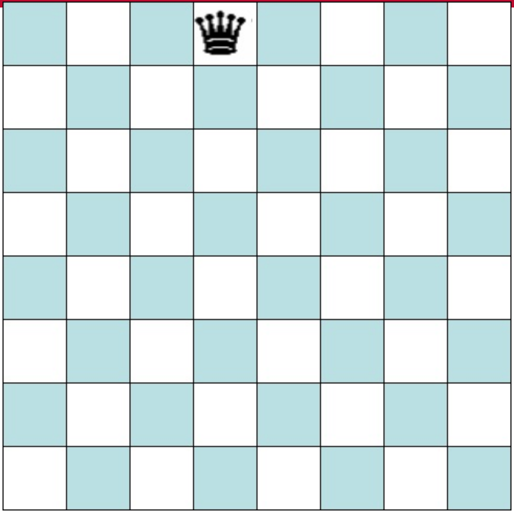
- Cannot put two Queens in same column: For all $i \neq j$, $Q_i \neq Q_j$

  or All-Diff$(Q_1, Q_2, \ldots, Q_N)$

- Diagonal constraints: for all $i \neq j$,

  $|Q_i - Q_j| \neq |i - j|$

Example: N-Queens



Solving CSPs: CSP as a Search Problem

A CSP could be formulated as a search problem:

- **Initial State**: Empty assignment.
- **Successor Function**: Assigned values to an unassigned variable.
- **Goal Test**:
  (1) The assignment is complete
  (2) No constraints is violated.

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CSP Backtracking Search - Intuition

CSPs do NOT require finding a path (to a goal). They only need the **configuration** of the goal state.
CSPs are best solved by a specialized version search called **Backtracking Search**.

Key Intuitions:

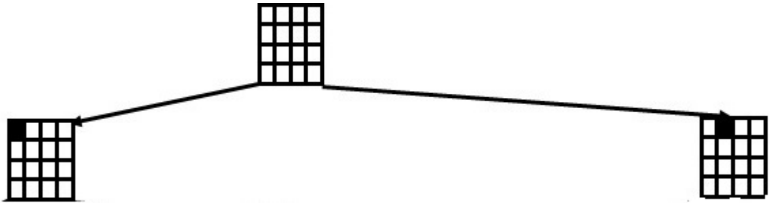
- Searching through the space of partial assignments, rather than paths.
- Decide on a suitable value for one variable at a time.
  Order in which we assign the variables does not matter.
- If a constraint is falsified during the process of partial assignment, immediately reject the current partial assignment.

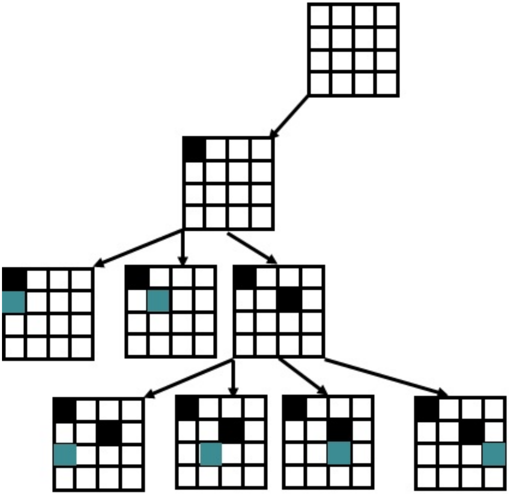
CSP Backtracking Search - Intuition

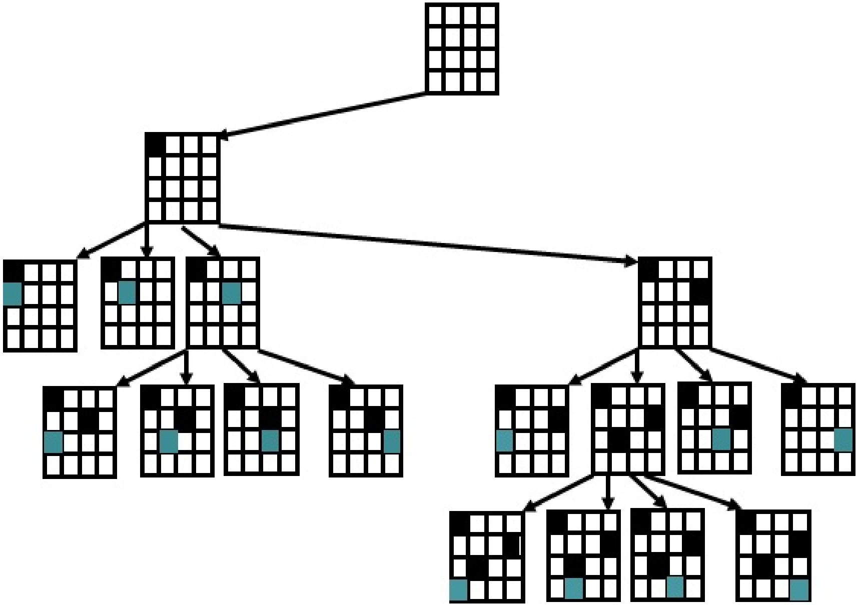
CSP Search Tree:

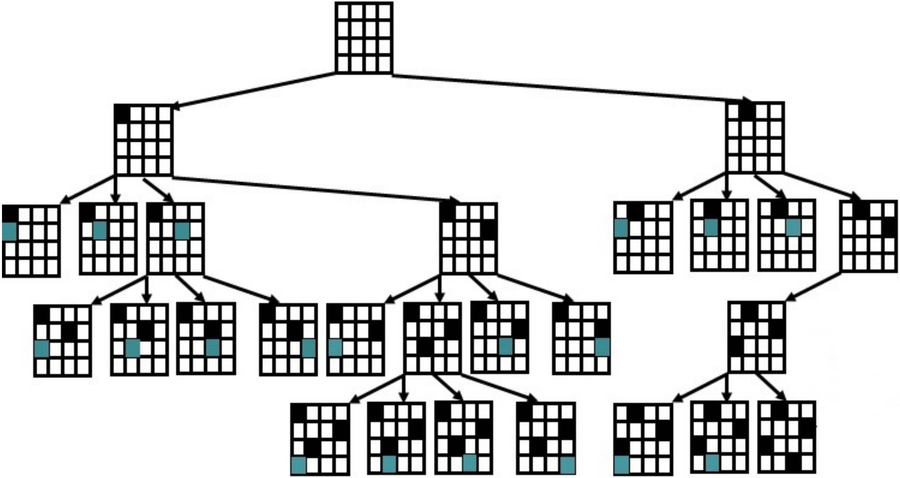
- **Root**: Empty Assignment.
- **Children** of a node: all possible value assignments for a particular unassigned variable.
- The tree **stops descending** if an assignment violates a constraint.
- **Goal Node**:
  (1) The assignment is complete
  (2) No constraints is violated.

Draw the CSP search tree for 4-Queens.









We will apply a **recursive** implementation:

- If all variables are set, print the solution and terminate.
- Otherwise:
  - Pick an unassigned variable V and assign it a value.
  - Test the constraints corresponding with V and all other variables of them are assigned.
  - If a constraint is unsatisfied, return (backtrack).
  - Otherwise, go one lever deeper by invoking a recursive call.

Backtracking Search: The Algorithm

```
def BT(Level):
   if all Variables assigned
      PRINT Value of each Variable
      EXIT or RETURN                     # EXIT for only one solution
                                         # RETURN for more solutions
   V := PickUnassignedVariable()
   Assigned[V] := TRUE
   for d := each member of Domain(V)    # the domain values of V
      Value[V] := d
      ConstraintsOK := TRUE
      for each constraint C such that (i) V is a variable of C and
                                     (ii) all other variables of C are assigned:
           if C is not satisfied by the set of current assignments:
                   ConstraintsOK := FALSE
      if ConstraintsOk == TRUE:
           BT(Level+1)
    Assigned[V] := FALSE     # UNDO as we have tried all of V's values
    RETURN
```