CSC384
Knowledge Representation
Part 1

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Credits

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Introduction

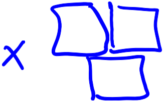
What is Knowledge Representation and Reasoning (KR&R)?
Symbolic encoding of propositions believed by some agent and their manipulation to produce propositions that are believed by the agent but not explicitly stated.

Why KR&R:

- Large amounts of knowledge are used to understand the world around us.
- **Reasoning** provides compression in the knowledge we need to store.
- **Without** reasoning we would have to store an infeasible amount of information:
  Example: Elephants can't fit into teacups, Elephants can't fit into cars, instead of just knowing that larger objects can't fit into smaller objects.

Introduction

- **Information:**
  (1) Block A is above block B;
  (2) Block B is above block C.



- **Query:** Is A above C?

Given the information, human can easily draw the conclusion.
How can a **machine** do the same?

Introduction

- Tony, Mike, and John are members of the Alpine Club.
- Every member of the Alpine Club who is not a skier is a mountain climber.
- Mountain climbers do not like rain, and anyone who does not like snow is not a skier.
- Mike dislikes whatever Tony likes, and likes whatever Tony dislikes.
- Tony likes rain and snow.
- Is there a member of the Alpine Club who is a mountain climber but not a skier?

Logical representations

- are mathematically precise; thus it's possible to analyze their limitations, properties, and complexity of inferences.
- are formal languages; thus computer programs can manipulate sentences in the language.
- typically, have well-developed proof theories: formal procedures for reasoning to produce new sentences.

In this module we will study **First-Order logic (FOL)**, and a reasoning mechanism called **resolution** that operates on First-Order logic.

Propositional Variable: A variable which takes only **True** or **False** as values.

The set of all propositional formulas is defined recursively as follows:

- Every propositional variable is a propositional formula;
- If φ is a propositional formula, then so is $\neg\varphi$;
- If φ_1 and φ_2 are propositional formulas, then so are
  - $\varphi_1 \wedge \varphi_2$ (**Conjunction**);
  - $\varphi_1 \vee \varphi_2$ (**Disjunction**);
  - $\varphi_1 \to \varphi_2$ (**Implication**); $\to \neg\varphi_1 \vee \varphi_2$
  - $\varphi_1 \leftrightarrow \varphi_2$ (**Bi-implication**).

Truth Assignment: A function τ from the propositional variables into the set of truth values $\{T, F\}$.

p q $\tau(p) = T$ / F $\tau(q) = T$ / F

Let τ be a truth assignment. The extension $\bar{\tau}$ of τ assigns either T or F to every formula and is defined as follows:

- If $A = x$, where x is a variable, then $\bar{\tau}(A) = \tau(x)$.
- $\bar{\tau}(\neg A) = T$ iff $\bar{\tau}(A) = F$;
- $\bar{\tau}(A \wedge B) = T$ iff $\bar{\tau}(A) = T$ and $\bar{\tau}(B) = T$;
- $\bar{\tau}(A \vee B) = T$ iff $\bar{\tau}(A) = T$ or $\bar{\tau}(B) = T$;
- $\bar{\tau}(A \to B) = F$ iff $\bar{\tau}(A) = T$ and $\bar{\tau}(B) = F$.

Example: Let $V = \{p, r, q\}$ be a set of propositional variables and $\tau_1 : V \to \{T, F\}$ and $\tau_2 : V \to \{T, F\}$ be two truth assignments s.t.:

- $\tau_1(p) = T$, $\tau_1(q) = F$, $\tau_1(r) = F$.
- $\tau_2(p) = F$, $\tau_2(q) = T$, $\tau_2(r) = F$.

Then

$\bar{\tau}_1((\overbrace{\neg p \wedge q}^{F}) \to r) = T$

$\bar{\tau}_2((\overbrace{\underbrace{\neg p}_{T} \wedge \underbrace{q}_{T}}^{T}) \to \overset{F}{r}) = F$

A truth assignment τ **satisfies** a formula A iff $\bar{\tau}(A) = T$.
τ satisfies a set Φ of formulas iff τ satisfies all formula in Φ.

A set Φ of formulas is satisfiable iff some truth assignment τ satisfies Φ.
Otherwise, Φ is unsatisfiable.

Example:

$$\Phi_1 = \{\underbrace{\overset{\to F}{r} \to (p \wedge q)}_{T}, \underbrace{\neg p}_{T}\}$$

$\tau(r) = F$ $\tau(p) = F$ $\tau(q) = T$

$$\Phi_2 = \{\underset{T}{r} \to \overbrace{\underset{F}{(p \wedge q)}}^{F}, \underset{T}{r} \wedge \underset{T}{\neg p}\} \Rightarrow p \to F$$

unsatisfiable

A formula A is a **logical consequence** of Φ (denoted by $\Phi \models A$) iff for every truth assignment τ, if τ satisfies Φ, then τ satisfies A.

Example: Let $\Phi = \{r \to ((p \land q) \lor s), r \land p\}$.

Then $\Phi \models q \lor s$

$\Phi \not\models q$

$\tau(r) = T$

$\tau(p) = T$

$\tau(s) = T$

$\tau(q) = F$

Limitations of Propositional Language

- **Only Boolean variables**: Without non-Boolean variables **cross references between individuals** in statements are **impossible**.
  Example: 'If a person has a sibling and that sibling has a child, then the person is an aunt or an uncle.'
  S: a person has a sibling.
  C: a sibling has a child.
  A: a person is an aunt or an uncle.

  $S \wedge C \rightarrow A$

  This approach doesn't work:
  person in S and A are not related.
  sibling in S and C are not related.

- **No quantifiers:** To state a property for all (or some) members of the domain we have to **explicitly list** them.
  Example: 'Every member of the Alpine Club who is not a skier is a mountain climber'

For **first-order logic** following components are required:

- A set V of variables.
- A set F of function symbols.
- A set P of predicate (relation) symbols.

- **Functions** and **variables** are used to construct terms. Terms denote elements of the domain.

father_of(Tom)

- **Predicates** are defined over terms. Like(Tom, rain)
  Atomic formulas denote properties and relations that hold about the elements in the domain.

- **Predicates** and **terms** are used to construct formulas.
  Other formulas generate more complex assertions by composing atomic formulas.

A set $\mathcal{L}$ of **function** and **predicate symbols** is called a first-order vocabulary.

Let $\mathcal{L}$ be a set of function and predicate symbols.

1. Every variable is a term.
2. If f is an n-ary function symbol in $\mathcal{L}$ and $t_1, t_2, ..., t_n$ are $\mathcal{L}$-terms, then $f(t_1, t_2, ..., t_n)$ is a $\mathcal{L}$-term.

Note: 0-ary functions symbols are called **constant symbols**.

Example:

$g, c_1, c_2, f \in \mathcal{L}$

binary

0-ary function

binary

$f(c_1, c_2)$

$g(x, c_1)$

$f(x, g(x, y))$

First-Order Logic: Syntax

Let $\mathcal{L}$ be a vocabulary. The set of first-order $\mathcal{L}$-formulas is defined recursively:

1. Atomic Formula: $P(t_1, t_2, \ldots, t_n)$, where P is an n-ary predicate symbol in $\mathcal{L}$ and $t_1, t_2, \ldots, t_n$ are $\mathcal{L}$-terms.

2. Negation: $\neg f$, where f is a $\mathcal{L}$-formula.

3. Conjunction: $f_1 \wedge f_2 \wedge \ldots \wedge f_n$, where $f_1, f_2, \ldots, f_n$ are $\mathcal{L}$-formulas.

4. Disjunction: $f_1 \vee f_2 \vee \ldots \vee f_n$, where $f_1, f_2, \ldots, f_n$ are $\mathcal{L}$-formulas.

5. Implication: $f_1 \to f_2$, where f_1, f_2 are $\mathcal{L}$-formulas.

$f_1 \leftrightarrow f_2$

6. Existential: $\exists x f$, where x is a variable and f is a $\mathcal{L}$-formula.

$\exists x \in \mathbb{N}, f$

7. Universal: $\forall x f$, where x is a variable and f is a $\mathcal{L}$-formula.

$\forall x \in \mathbb{N}, f$

Converting English to First-Order Language

- **Individuals**: Constants (0-ary Functions)
  - **tony, mike, john rain, snow**

$\mathcal{L}$

- **Types**: Unary Predicates
  - $AC(x)$: x belongs to Alpine Club.
  - $S(x)$: x is a skier.
  - $C(x)$: x is a mountain climber.
- **Relationships**: Binary Predicates
  - $L(x, y)$: x likes y.

- **Basic Facts**:

  - Tony, Mike, and John belong to the Alpine Club:
    $AC(\textbf{tony}), AC(\textbf{mike}), AC(\textbf{john})$

    ~~$\forall x, x \in AC \rightarrow$~~

  - Tony likes rain and snow:
    $L(\textbf{tony, rain}), L(\textbf{tony, snow})$

- **Complex Facts**:

  - Every member of the Alpine Club who is not a skier is a mountain climber.

    $\forall x [AC(x) \wedge \neg S(x) \rightarrow C(x)]$

  - Mountain climbers do not like rain, and anyone who does not like snow is not a skier.

$\forall x [C(x) \rightarrow \neg L(x, rain)] \wedge \forall x [\neg L(x, snow) \rightarrow \neg S(x)]$

– Mike dislikes whatever Tony likes, and likes whatever Tony dislikes.

$\forall x [L(tony, x) \rightarrow \neg L(mike, x)] \wedge$

$\forall x [\neg L(tony, x) \rightarrow L(mike, x)]$

– Is there a member of the Alpine Club who is a mountain climber but not a skier?

$\exists x [AC(x) \wedge C(x) \wedge \neg S(x)]$

Like variables in programming languages, the variables in FOL have a scope which is determined by the quantifiers.

Lexical scope for variables:

$Animal(x) \wedge \exists x \underbrace{[Human(x) \vee Women(x)]}$.

$\exists x \underbrace{[Animal(x) \rightarrow \neg Human(x)]} \wedge \exists x \underbrace{[Human(x) \vee Women(x)]}$

First-Order Logic: Semantic

- In the **propositional logic**, a **truth assignment** provides meaning to a formula.
- In **FOL** we can talk about **(non-Boolean) individuals and elements**.
  So the simple universe of truth values is not rich enough to provide a suitable interpretation for FOL formulas.
- We need more **more complicated objects** to give meaning to formulas and terms.
- These objects are called **structures**.

Let $\mathcal{L}$ be a first-order vocabulary. An $\mathcal{L}$-**structure** $\mathcal{M}$ consists of the following:

1. A **nonempty set** M called the universe (domain) of discourse.
2. For each n-ary **function symbol** $f \in \mathcal{L}$, an associated function $f^{\mathcal{M}} : M^n \to M$.
   Note: If $n = 0$, then f is a constant symbol and $f^{\mathcal{M}}$ is simply an element of M.
   $f^{\mathcal{M}}$ is called the **extension** of the function symbol f in $\mathcal{M}$.
3. For each n-ary **predicate symbol** $P \in \mathcal{L}$, an associated relation $P^{\mathcal{M}} \subseteq M^n$.
   $P^{\mathcal{M}}$ is called the **extension** of the predicate symbol P in $\mathcal{M}$.

Blocks World:

Suppose $\mathcal{L}_{BW}$ includes the following symbols:

- **Function Symbols:**
  - $under(x)$: the block immediately under x if x is not on table; x itself otherwise.
- **Predicate Symbols:**
  - $on(x, y)$: x is place (directly) on y.
  - $above(x, y)$: x is above y.
  - $clear(x)$: no blocks are above x.
  - $ontable(x)$: no blocks are under x.

Suppose $\mathcal{L}_{BW}$ includes the following symbols:

- **Function Symbols:**
  - $under(x)$: the block immediately under x if x is not on table; x itself otherwise.
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$\mathcal{M}_1$ is a $\mathcal{L}_{BW}$-structure such that:

$M_1 = \{A, B, C, D\}$

$on^{\mathcal{M}_1} = \{\langle A, B\rangle, \langle B, C\rangle\}$

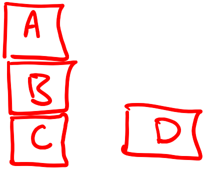
$above^{\mathcal{M}_1} = \{\langle A, B\rangle, \langle B, C\rangle, \langle A, C\rangle\}$

$clear^{\mathcal{M}_1} = \{A, D\}$

$ontable^{\mathcal{M}_1} = \{C, D\}$

$under^{\mathcal{M}_1}(A) = B$, $under^{\mathcal{M}_1}(B) = C$,

$under^{\mathcal{M}_1}(C) = C$, $under^{\mathcal{M}_1}(D) = D$



Suppose $\mathcal{L}_{BW}$ includes the following symbols:

- **Function Symbols:**
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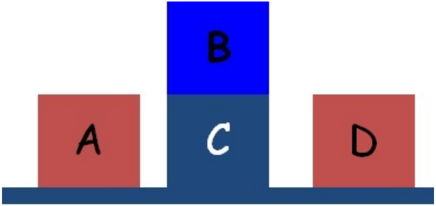
  - $above(x, y)$: x is above y.
  - $clear(x)$: no blocks are above x.
  - $ontable(x)$: x is placed on the table.

Represent the following configuration by a $\mathcal{L}_{BW}$-structure.



M_2:

$M_2 = \{A, B, C, D\}$

$On^{M_2} = \{\langle B, C\rangle\}$

$above^{M_2} = \{\langle B, C\rangle\}$

$Clear^{M_2} = \{A, B, D\}$

$Ontable^{M_2} = \{A, C, D\}$

$under^{M_2}(A) = A$

$under^{M_2}(B) = C$

$under^{M_2}(C) = C$ $\quad$ $under^{M_2}(D) = D$

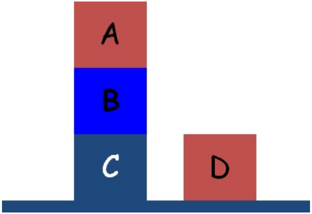
Semantic of First-Order Logic: Intuition

Every $\mathcal{L}$-formula becomes either true or false when interpreted by an $\mathcal{L}$-structure $\mathcal{M}$.

That is, the truth value of a first-order formulas A is evaluated w.r.t to a first-order structure $\mathcal{M}$:

- Terms (variables and functions) of a formula denote elements of the domain.
  So every term in A must correspond with an element of the universe of $\mathcal{M}$.
- Atomic formulas denote properties and relations that hold about the elements in the domain.
  $P(t_1, ..., t_n)$ is true in $\mathcal{M}$ if $t_1, ..., t_n$ are related to each other by $P^{\mathcal{M}}$.
- Other formulas generate more complex assertions by composing atomic formulas.
  Their truth is dependent on the truth of the atomic formulas in them.

Let $\mathcal{M}$ be a structure and X be a set of variables. An **object assignment** σ for $\mathcal{M}$ is a **mapping** from variables in X to the universe of $\mathcal{M}$.



$$X = \{v_1, v_2, v_3, v_4\}$$

$$\sigma(v_1) = D, \qquad \sigma(v_2) = C$$
$$\sigma(v_3) = B, \qquad \sigma(v_4) = A$$

$\sigma'(v_1) = C \quad \sigma'(v_2) = C$

$\sigma'(v_3) = A \quad \sigma'(v_4) = B$

Remember the recursive definition of term:
Let $\mathcal{L}$ be a set of function and predicate symbols.

1. Every variable x is a term.
2. If f is an n-ary function symbol in $\mathcal{L}$ and $t_1, t_2, ..., t_n$ are $\mathcal{L}$-terms, then $f(t_1, t_2, ..., t_n)$ is a $\mathcal{L}$-term.

Let $\mathcal{L}$ be a vocabulary and $\mathcal{M}$ be an $\mathcal{L}$-structure.
The extension $\bar{\sigma}$ of σ is defined recursively:

1. for every variable x, $\bar{\sigma}(x) = \sigma(x)$;
2. for every function symbol $f \in \mathcal{L}$, $\bar{\sigma}(f(t_1, ..., t_n)) = f^{\mathcal{M}}(\bar{\sigma}(t_1), ..., \bar{\sigma}(t_n))$.

Let $\mathcal{L}$ be a vocabulary and $\mathcal{M}$ be an $\mathcal{L}$-structure.
The extension $\bar{\sigma}$ of σ is defined recursively:

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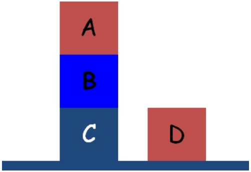

$$under^{\mathcal{M}}(A) = B \qquad under^{\mathcal{M}}(B) = C$$
$$under^{\mathcal{M}}(C) = C \qquad under^{\mathcal{M}}(D) = D$$

$X = \{v_1, v_2, v_3, v_4\}$
$\sigma(v_1) = D, \qquad \sigma(v_2) = C$
$\sigma(v_3) = B, \qquad \sigma(v_4) = A$

$\bar{\sigma}(under(under(v_4))) = under^{M}(\underbrace{\bar{\sigma}(under(v_4))}_{B}) = under^{M}(B) = C$

$$\bar{\sigma}(\text{under}(v_4)) = \text{under}^m(\underbrace{\bar{\sigma}(v_4)}_{A}) = \text{under}^m(A) = B$$

$$\bar{\sigma}(v_4) = \sigma(v_4) = A$$

$$\sigma_2(v_4) = D$$

$$\bar{\sigma}_2(\text{under}(\text{under}(v_4))) = D$$

First-Order Logic Semantic: Models (Interpretations)

For an $\mathcal{L}$-formula C, $\mathcal{M} \models C[\sigma]$ ($\mathcal{M}$ **satisfies** C under σ, or $\mathcal{M}$ is a **model** of C under σ) is defined recursively on the structure of C as follows (assuming A, B are $\mathcal{L}$-formulas):

$\mathcal{M} \models P(t_1, ..., t_n)[\sigma]$	iff	$\langle \bar{\sigma}(t_1), ..., \bar{\sigma}(t_n) \rangle \in P^{\mathcal{M}}$.
$\mathcal{M} \models (s = t)[\sigma]$	iff	$\bar{\sigma}(s) = \bar{\sigma}(t)$.
$\mathcal{M} \models \neg A[\sigma]$	iff	$\mathcal{M} \not\models A[\sigma]$.
$\mathcal{M} \models (A \vee B)[\sigma]$	iff	$\mathcal{M} \models A[\sigma]$ or $\mathcal{M} \models B[\sigma]$.
$\mathcal{M} \models (A \wedge B)[\sigma]$	iff	$\mathcal{M} \models A[\sigma]$ and $\mathcal{M} \models B[\sigma]$.
$\mathcal{M} \models (\forall x A)[\sigma]$	iff	$\mathcal{M} \models A[\sigma(m/x)]$ for all $m \in M$.
$\mathcal{M} \models (\exists x A)[\sigma]$	iff	$\mathcal{M} \models A[\sigma(m/x)]$ for some $m \in M$.

First-Order Logic Semantic: Models (Interpretations)

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$\mathcal{M} \models (s = t)[\sigma]$	iff	$\bar{\sigma}(s) = \bar{\sigma}(t)$.
$\mathcal{M} \models \neg A[\sigma]$	iff	$\mathcal{M} \not\models A[\sigma]$.
$\mathcal{M} \models (A \vee B)[\sigma]$	iff	$\mathcal{M} \models A[\sigma]$ or $\mathcal{M} \models B[\sigma]$.
$\mathcal{M} \models (A \wedge B)[\sigma]$	iff	$\mathcal{M} \models A[\sigma]$ and $\mathcal{M} \models B[\sigma]$.
$\mathcal{M} \models (\forall x A)[\sigma]$	iff	$\mathcal{M} \models A[\sigma(m/x)]$ for all $m \in M$.
$\mathcal{M} \models (\exists x A)[\sigma]$	iff	$\mathcal{M} \models A[\sigma(m/x)]$ for some $m \in M$.

Note: $\sigma(m/x)$ is an object assignment function exactly like σ, but maps the variable x to the individual $m \in M$. That is:

For $y \neq x : \sigma(m/x)(y) = \sigma(y)$

For x: $\sigma(m/x)(x) = m$

Let $\mathcal{M}_3$ be a structure such that:
$M_3 = \{A, B, C, D\}$
$on^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle\}$
$above^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle, \langle A, C\rangle\}$
$clear^{\mathcal{M}_3} = \{A, D\}$
$ontable^{\mathcal{M}_3} = \{C, D\}$

Does $\mathcal{M}_3$ satisfy
$\forall x \forall y(on(x, y) \rightarrow above(x, y))$ ✓

$x = A$: $y = A$ ✓, $y = B$ ✓, $y = C$ ✓, $y = D$ ✓

$x = B$: $y = A$ ✓, $y = B$ ✓, $y = C$ ✓, $y = D$ ✓

$x = C$: $y = A$ ✓, $y = B$ ✓, $y = C$ ✓, $y = D$ ✓

$x = D$: $y = A$ ✓, $y = B$ ✓, $y = C$ ✓, $y = D$ ✓

Let $\mathcal{M}_3$ be a structure such that:
$M_3 = \{A, B, C, D\}$
$on^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle\}$
$above^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle, \langle A, C\rangle\}$
$clear^{\mathcal{M}_3} = \{A, D\}$
$ontable^{\mathcal{M}_3} = \{C, D\}$

Does $\mathcal{M}_3$ satisfy
$\forall x \forall y (above(x, y) \rightarrow on(x, y))$ ✗

$x = A$ $y = C$ ✗

$\langle A, C\rangle \in above^{\mathcal{M}_3}$

$\langle A, C\rangle \notin on^{\mathcal{M}_3}$

Let $\mathcal{M}_3$ be a structure such that:
$M_3 = \{A, B, C, D\}$
$on^{\mathcal{M}_3} = \{\langle A, B\rangle, \underline{\langle B, C\rangle}\}$
$above^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle, \langle A, C\rangle\}$
$clear^{\mathcal{M}_3} = \{A, \underline{D}\}$
$ontable^{\mathcal{M}_3} = \{C, D\}$

Does $\mathcal{M}_3$ satisfy
$\forall x \exists y (\underline{clear(x)} \vee On(y, x))$ ✓

$x = A$ $\quad y = A, B, C, D$ ✓

$x = B$ $\quad y = A$ ✓

$x = C$ $\quad y = B$ ✓

$x = D$ $\quad y = A, B, C, D$ ✓

Let $\mathcal{M}_3$ be a structure such that:
$M_3 = \{A, B, C, D\}$
$on^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle\}$
$above^{\mathcal{M}_3} = \{\langle A, B\rangle, \langle B, C\rangle, \langle A, C\rangle\}$
$clear^{\mathcal{M}_3} = \{A, D\}$
$ontable^{\mathcal{M}_3} = \{C, D\}$

Does $\mathcal{M}_3$ satisfy
$\exists y \forall x (clear(x) \vee On(y, x))$ X

$y = A$ $x = C$ X

$y = B$ $x = B$ X

$y = C$ $x = C$ X

$y = D$ $x = C$ X

An occurrence of x in A is **bounded** iff it is in a sub-formula of A of the form $\forall x B$ or $\exists x B$. Otherwise the occurrence is **free**.

Example:

$P(x) \wedge \exists x[P(x) \vee Q(x)]$

free

bounded

In a structure $\mathcal{M}$, formulas with **free variables** might be **true for some** object assignments to the free variables and **false for others**.

Example: Consider the formula $\overbrace{P(x, y) \wedge P(y, x)}^{A}$ and the following structure $\mathcal{M}$:

$M = \{a, b\} \qquad P^{\mathcal{M}} = \{\langle a, a\rangle\}$

$\sigma_1(x) = a \quad \sigma_1(y) = a \qquad M \models A[\sigma_1]$

$\sigma_2(x) = a \quad \sigma_2(y) = b \qquad M \not\models A[\sigma_2]$

First-Order Logic Semantic: Models

A formula A is **closed** if it contains no free occurrence of a variable.
A **closed formula** is called a **sentence**.
Example:
$P(x) \wedge \exists x[P(x) \vee Q(x)]$. ✗

$\forall x P(x) \wedge \exists x[P(x) \vee Q(x)]$ ✓

If σ and σ' agree on the **free variables** of A, then $\mathcal{M} \models A[\sigma]$ iff $\mathcal{M} \models A[\sigma']$.
Proof: Structural induction on A.

Corollary: If A is a **sentence**, then for any object assignments σ and σ',

$$\mathcal{M} \models A[\sigma] \qquad \text{iff} \qquad \mathcal{M} \models A[\sigma']$$

So, if A is a **sentence** (no free variables), σ is **irrelevant** and we omit mention of σ and simply write $\mathcal{M} \models A$.

Let Φ be a **set of sentences**.

- $\mathcal{M}$ satisfies Φ (denoted by $\mathcal{M} \models \Phi$) if for **every** sentence $A \in \Phi$, $\mathcal{M} \models A$.
- If $\mathcal{M} \models \Phi$, we say $\mathcal{M}$ is a model of Φ.
- We say that Φ is satisfiable if there is a structure $\mathcal{M}$ such that $\mathcal{M} \models \Phi$.

Unintended Models: Example

Let Φ_1 be a set containing the following sentences
($\boldsymbol{c_1}, \boldsymbol{c_2}$ are constant symbols, we use **bold** font to distinguish constant symbols from variables):

- $on(\boldsymbol{c_1}, \boldsymbol{c_2})$
- $clear(\boldsymbol{c_1})$
- $above(\boldsymbol{c_1}, \boldsymbol{c_2})$

Consider a model of Φ_1 with **size three** (i.e., the size of the domain of the model is three).

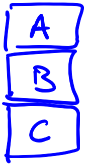
$M_1 = \{A, B, C\}$

$\boldsymbol{c_1}^{\mathcal{M}_1} = A \qquad \boldsymbol{c_2}^{\mathcal{M}_1} = B$

$on^{\mathcal{M}_1} = \{\langle A, B\rangle, \langle B, C\rangle\}$

$clear^{\mathcal{M}_1} = \{A, C\}$

$above^{\mathcal{M}_1} = \{\langle A, B\rangle\}$



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Eliminating Unintended Models: Example

Let Φ_2 be a set containing the following sentences($\boldsymbol{c_1, c_2}$ are constant symbols):

- $\forall x(clear(x) \rightarrow \neg\exists y(on(y,x)))$
- $\forall x \forall y(on(x,y) \rightarrow above(x,y))$
- $\forall x \forall y \forall z((above(x,y) \wedge above(y,z)) \rightarrow above(x,z))$
- $on(\boldsymbol{c_1, c_2})$
- $clear(\boldsymbol{c_1})$
- $above(\boldsymbol{c_1, c_2})$

Construct **two models** of Φ_2 with **size three** (i.e., the size of the domain of each model must be three).

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Example: is $\{\forall x(P(x) \rightarrow Q(x)), P(\mathbf{a}), \neg Q(\mathbf{a})\}$ satisfiable?