Heuristic Search

- Reading note: Chapter 4 in R&N covers heuristic search.
- 2nd Edition of R&N is (temporarily) available at: https://catalog.hathitrust.org/Record/004917484?type%5B%5D=all&lookfor%5B%5D=Artificial%20Intelligence%20Approach%20Modern%20ft=ft
- Also https://artint.info/2e/html/ArtInt2e.Ch3.S6.html

Heuristic functions

We can encode each notion of the “merit” of a state into a heuristic function, \( h(n) \).

A heuristic function maps a state onto an estimate of the cost to the goal from that state.

Can you think of examples of heuristics?

Heuristics are sensitive to the problem domain.

- Heuristic for planning a path through a maze?
- For solving a Rubick’s cube?

Euclidean distance as \( h(s) \)

Say we want to plan a path from Arad to Bucharest, and we know the straight line distance from each city to our goal. This lets us plan our trip by picking cities at each time point that minimize the distance to our goal (or maximize our heuristic).
Best first (greedy) search

If we only use $h(n)$ to guide our search, the search strategy is called Greedy or Best First Search.

How can it possibly go wrong??

In red is the path we selected. In green is the shortest path between Arad and Bucharest.

What happened?

Modifying the search

How to avoid the mistake?
Modifying the search

How to avoid the mistake?

Take into account the cost of getting to the node as well as our estimate of the cost of getting to the goal from the node.

Define an evaluation function \( f(n) \):

\[
f(n) = g(n) + h(n)
\]

- \( g(n) \) is the cost of the path to node \( n \).
- \( h(n) \) is the heuristic estimate of the cost of achieving the goal from \( n \).

Always expand the node with lowest \( f \)-value on Frontier.

The \( f \)-value, \( f(n) \), is an estimate of the cost of getting to the goal via the node (path) \( n \).

I.e., we first follow the path \( n \) then we try to get to the goal. \( f(n) \) estimates the total cost of such a solution.

What will happen here??

Terminate only after we have REMOVED the goal from the Frontier.
What happens when we visit nodes twice?

If A* discovers a lower cost path through a state as it is searching, it should update the order of that state on the Frontier, based on the lower path cost.

Is A* Guaranteed Optimal?

Properties of A* depend on conditions on \( h(n) \)

- To achieve completeness, optimality, and desirably time and space complexity with A* search, we must put some conditions on the heuristic function \( h(n) \) and the search space.
Condition on \( h(n) \): Admissible

- Assume each transition due to an action \( a \) has cost \( \geq \epsilon > 0 \).
- Let \( h^*(n) \) be the cost of an optimal path from \( n \) to a goal node \( (\infty \text{ if there is no path}) \). Then an admissible heuristic satisfies the condition:

\[
h(n) \leq h^*(n)
\]

an admissible heuristic never over-estimates the cost to reach the goal, i.e., it is optimistic.

- Hence \( h(g) = 0 \), for any goal node \( g \)
- Also \( h^*(n) = \infty \) if there is no path from \( n \) to a goal node

Search animations: Pac Man

https://www.youtube.com/watch?v=2XpjA4GWzY

Problem!

\[
\begin{align*}
\text{A} & \quad 1 \\
\text{B} & \quad 2 \\
\text{C} & \quad 9 \\
\text{D} & \quad 6 \\
\end{align*}
\]

\[
\begin{align*}
h(A) &= 8 \\
h(B) &= 3 \\
h(C) &= 7 \\
h(D) &= 0
\end{align*}
\]

Back to admissibility

Which heuristics are admissible for the 8 puzzle?

- \( h(n) = \) number of misplaced tiles
- \( h(n) = \) total Manhattan distance between tile locations in \( S \) and goal locations in \( G \)
- \( h(n) = \min (2, h^*(n)) \)
- \( h(n) = h^*(n) \)
- \( h(n) = \max (2, h^*(n)) \)
- \( h(n) = 0 \)

Record your answers at https://forms.gle/n7aWKFT4TSy6vDuJ9
Admissible heuristics

Say for the 8-puzzle:

- $h_1(n) = \text{number of misplaced tiles}$
- $h_2(n) = \text{total Manhattan distance}$
  (i.e., no. of squares from desired location of each tile)

$\bullet \ h_1(S) = \ ? \ 8$
$\bullet \ h_2(S) = \ ? \ 3+1+2+2+2+3+3+2 = 18$

Which heuristic might be preferable, and why?

How to build a heuristic?

A useful technique is to simplify a problem when building heuristics, and to let $h(n)$ be the cost of reaching the goal in the easier problem.

For example, in the 8-Puzzle you can only move a tile from square A to B if A is adjacent (left, right, above, below) to B and B is blank.

We can relax some of these conditions and:
1. allow a move from A to B if A is adjacent to B (i.e. we can ignore whether or not position is blank),
2. allow a move from A to B if B is blank (i.e. we can ignore adjacency),
3. allow all moves from A to B (ignore both conditions).

How to build a heuristic?

- #3 leads to the misplaced tiles heuristic.
  - To solve the puzzle, we need to move each tile into its final position.
  - Number of moves = number of misplaced tiles.
  - Clearly $h(n) = \text{number of misplaced tiles} \leq h^*(n)$ the cost of an optimal sequence of moves from $n$.

- #1 leads to the Manhattan distance heuristic.
  - To solve the puzzle we need to slide each tile into its final position.
  - We can move vertically or horizontally.
  - Number of moves = sum over all of the tiles of the number of vertical and horizontal slides we need to move that tile into place.
  - Again $h(n) = \text{sum of the Manhattan distances} \leq h^*(n)$
    - in a real solution we need to move each tile at least that far and we can only move one tile at a time.

Admissible heuristics make for optimal search

Why?
Admissible heuristics make for optimal search

Why?

• Say we have an optimal path to \( n_{\text{goal}} \) with cost \( g(n_{\text{goal}}) \).
• Let \( n'_{\text{goal}} \) be a sub-optimal path, meaning \( g(n'_{\text{goal}}) > g(n_{\text{goal}}) \).
• Let \( n'' \) be any sub-path of the optimal path on the Frontier.

Is it possible for the path to \( n'_{\text{goal}} \) to be explored before the path to \( n_{\text{goal}} \)?

• No! Because \( f(n_{\text{goal}}) < f(n'_{\text{goal}}) \)
• Also \( f(n'') \leq f(n_{\text{goal}}) \), because our heuristic is admissible.
• So, \( f(n'') < f(n'_{\text{goal}}) \)

Meaning sub-paths on the optimal path to \( n_{\text{goal}} \) will be explored before any sub-optimal path to the goal!

Weighted A*

• Weighted A* defines an evaluation function \( f(n) \):
  \[
  f(n) = g(n) + \varepsilon h(n)
  \]
  
  - \( \varepsilon > 1 \) introduces a bias towards states that are closer to the goal.
  - \( \varepsilon = 1 \) generates a provably optimal solution (assuming admissible heuristic).

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  - \( \varepsilon = 1 \) generates a provably optimal solution (assuming admissible heuristic).

• Search is typically orders of magnitude faster
• Path that is discovered may be sub-optimal (by factor that depends on \( \varepsilon \))
Anytime A*

• Weighted A* can be used to construct an anytime algorithm:
  – Find the best path for a given ε
  – Reduce ε and re-plan

\[ \varepsilon = 2 \]
13 node expansions  
Solution length: 12

\[ \varepsilon = 1.5 \]
15 node expansions  
Solution length: 12

\[ \varepsilon = 1 \]
20 node expansions  
Solution length: 10

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Effect of Heuristic Functions

• What portion of the state space will be explored by UCS? A*? Greedy search? Weighted A*?

Stronger condition on h(n): Monotonic (or consistent)

• Stronger condition than admissibility

• A monotone heuristic satisfies the condition
  \[ h(n_1) \leq c(n_1, a, n_2) + h(n_2) \]

• Note that there might more than one transition (action) that joins n1 and n2, and the inequality must hold for all of them.

• If h(n) is admissible and monotonic, search will be both optimal and not "locally" mislead.

Consistency implies Admissibility

Assume consistency: \[ h(n_1) \leq c(n_1, a, n_2) + h(n_2) \]

Prove admissible: \( h(n) \leq h^*(n) \)

If no path exists from n to a goal, \( h^*(n) = \infty \) and \( h(n) \leq h^*(n) \).
Let the path to from n to \( n_{goal} \) be an OPTIMAL path from n to a goal. Call the cost of this path \( h^*(n) \), and call the cost of each sub-path from \( n_i \) to \( n_{goal} \), \( h^*(n_i) \).
We will prove \( h(n) \leq h^*(n) \) by induction on the length of this optimal path.

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Question about monotonicity: https://forms.gle/EuTS6fJPr8nezuY27
Proof by Induction

Assume consistency: $h(n_1) \leq c(n_1, a, n_2) + h(n_2)$

Prove admissible: $h(n) \leq h^*(n)$

Base Case:

- $h(n_{goal}) = 0 \leq h^*(n_{goal}) = 0$
- $h(n_i) \leq c(n_i, a_i, n_{goal}) + h(n_{goal}) \leq c(n_i, a_i, n_{goal}) + h^*(n_{goal}) = h^*(n_i)$

Induction:

Assume $h(n_i) \leq h^*(n_i)$

- $h(n_i-1) \leq c(n_i-1, a_i-1, n_i) + h(n_i) \leq c(n_i-1, a_i-1, n_i) + h^*(n_i) = h^*(n_{i-1})$

Some consequences of Monotonicity

1. f-values of states in a path are non-decreasing.

   i.e. if $n_1$ and $n_2$ are nodes along a path, then $f(n_1) \leq f(n_2)$

   Proof: $f(n_1) = g(n_1) + h(n_1) = cost(path to n_1) + h(n_1)$
   $\leq g(n_1) + c(n_1, a, n_2) + h(n_2)$
   But $g(n_1) + c(n_1, a, n_2) + h(n_2) = g(n_2) + h(n_2) = f(n_2)$

   So $f(n_1) \leq f(n_2)$

2. If $n_2$ is expanded after $n_1$, then $f(n_1) \leq f(n_2)$.

   i.e. f-values of nodes that are expanded cannot decrease.
Some consequences of Monotonicity

2. If n2 is expanded after n1, then f(n1) ≤ f(n2).
   i.e. f-values of nodes that are expanded cannot decrease during the search.

   Why? When n1 was selected for expansion, n2 was either:
   1. Already on the frontier, meaning f(n1) ≤ f(n2). Otherwise we would have expanded n2 before n1.
   2. Added to the frontier as a result of n1’s expansion, meaning n2 and n1 lie along the same path. If this is the case, as we demonstrated on the prior slide, f(n1) ≤ f(n2).

Some consequences of Monotonicity

3. If node n has been expanded, every path with a lower f-value than n has already been expanded.

   f(of the first discovered path to n) = cost(of the first discovered path to n) + h(n).
   Likewise, f(of any other path to n) = cost(of any other path to n) + h(n).

   From the prior slide we know:
   f(of the first discovered path to n) ≤ f(of any other path to n).

   This means, by substitution:
   cost(of 1st discovered path to n) ≤ cost(of any other path to n)

   Hence, the first discovered path is the optimal one!

Some consequences of Monotonicity

4. The first time A* expands a node, it has found the minimum cost path to that node.

   f(of the first discovered path to n) = cost(of the first discovered path to n) + h(n).
   Likewise, f(of any other path to n) = cost(of any other path to n) + h(n).

   From the prior slide we know:
   f(of the first discovered path to n) ≤ f(of any other path to n).

   This means, by substitution:
   cost(of 1st discovered path to n) ≤ cost(of any other path to n)
Monotonic, Admissible A*

Complete?
YES. Consider a least cost path to a goal node

\[ \text{SolutionPath} = \langle \text{Start} \rightarrow n_1 \rightarrow \ldots \rightarrow G \rangle \] with cost \( c(\text{SolutionPath}) \).

--Since each action has a cost \( \geq \varepsilon > 0 \), there are only a finite number of paths
that have \( f \)-value < \( c(\text{SolutionPath}) \). None of these paths lead to a goal node
since \( \text{SolutionPath} \) is a least cost path to the goal.

--So eventually \( \text{SolutionPath} \), or some equal cost path to a goal must be
expanded.

Time and Space complexity?

--When \( h(n) = 0 \) for all \( n \), \( h \) is monotone (A* becomes uniform-cost search)!

--When \( h(n) > 0 \) for some \( n \) and still admissible, the number of nodes
expanded will be no larger than uniform-cost.

--Hence the same bounds as uniform-cost apply. (These are worst case
bounds). Still exponential complexity unless we have a very good \( h \)!

--In real world problems, we sometimes run out of time and memory. We
will introduce IDA* to address some memory issues, but IDA* isn’t very
good when many cycles are present.

Monotonic, Admissible A*

Optimal?
YES. As we saw, the first path to a goal node must be
optimal.

Cycle Checking?

We can use a simple implementation of cycle checking
(multiple path checking) - just reject all search nodes that
visit a state already visited by a previously expanded
node. We need keep only the first path to a state,
rejecting all subsequent paths.

Limitations of A* Search

- Observation: While A* may expand less of the
state space, it is still constrained by speed or
memory (many states are explored, on Frontier).

- Tools to address these problems:
  - IDA* (Iterative Deepening A*) - similar to
    Iterative Deepening Search.
  - Weighted A* - A* with an added weight, to bias
    exploration toward goal. We looked at this a bit
    last time!

IDA* - Iterative Deepening A*

Objective: reduce memory requirements for A*

- Like Iterative Deepening, but now the “cutoff” is the \( f \)-value \( (g+h) \)
  rather than the depth

- At each iteration, the cutoff value is the smallest \( f \)-value of any node
  that exceeded the cutoff on the previous iteration

- Avoids overhead associated with keeping a sorted queue of nodes,
  and the open list occupies only linear space.

- Two new parameters:
  - curBound (any node with a bigger \( f \)-value is discarded)
  - smallestNotExplored (the smallest \( f \)-value for discarded nodes in
    a round); when Frontier becomes empty, the search starts a new
    round with this bound.

  - To compute “smallestNotExplored” most readily, expand all nodes
    with \( f \)-value EQUAL to the \( f \)-limit.
IDA* Example: 8-Puzzle

\[ f(n) = g(n) + h(n) \]
\[ h(n) = \text{number of misplaced tiles} \]
\[ \text{blank tile is white} \]

0 + 4 = g(n) + h(n) = 4
Cutoff=4

1 + 6 = g(n) + h(n) = 7

IDA* Example: 8-Puzzle

\[ f(n) = g(n) + h(n) \]
\[ h(n) = \text{number of misplaced tiles} \]
\[ \text{blank tile is white} \]

Cutoff=4
IDA* Example: 8-Puzzle

\[ f(n) = g(n) + h(n) \]

\[ h(n) = \text{number of misplaced tiles} \]

Cutoff=4

Cutoff=6

8-Puzzle

\[ f(n) = g(n) + h(n) \]

\[ h(n) = \text{number of misplaced tiles} \]

Cutoff=6
8-Puzzle

\[ f(n) = g(n) + h(n) \]

\[ h(n) = \text{number of misplaced tiles} \]
**Comparing Iterative Deepening with A***
From Russell and Norvig

For 8-puzzle, average number of states expanded over 100 randomly chosen problems in which optimal path is length...

<table>
<thead>
<tr>
<th>Method</th>
<th>Steps</th>
<th>Nodes</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iterative Deepening</td>
<td>112</td>
<td>6,300</td>
<td>3.6 x 10^6</td>
</tr>
<tr>
<td>A* search using &quot;number of misplaced tiles&quot; as the heuristic</td>
<td>13</td>
<td>39</td>
<td>227</td>
</tr>
<tr>
<td>A* using &quot;Sum of Manhattan distances&quot; as the heuristic</td>
<td>12</td>
<td>25</td>
<td>73</td>
</tr>
</tbody>
</table>

**IDA* - Iterative Deepening A***

- Optimal?
- Complete?
- Time and Space Complexity?
- Cycle Checking?