CSC384
Constraint Satisfaction Problems
Part 3

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Constraint Propagation: Generalized Arc Consistency

A constraint $C(V_1, V_2, V_3, ..., V_n)$ is **GAC wrt** a variable V_i iff for **every domain value** of V_i, there exist domain values for $V_1, V_2, ..., V_{i-1}, V_{i+1}, ..., V_n$ that satisfy $C(V_1, V_2, V3, ..., V_n)$.

$C(V_1, V_2, V3, ..., V_n)$ is **GAC** iff it is GAC with respect to **all variables** in its scope.

A **CSP** is **GAC** if and only if **all of its constraints** are GAC.

Say we find a value d of variable V_i that is **not consistent** wrt a constraint: that is, there is **no assignments** to the other variables that satisfy the constraint when $V_i = d$:

- d is said to be **Arc Inconsistent**.
- We can **remove** d from the domain of V_i as this value cannot lead to a solution (much like Forward Checking, but more powerful).

Example: $C(X, Y) : X > Y$

$Dom[X] = \{\cancel{1}, 5, 11\}, Dom[Y] = \{3, 8, \cancel{15}\}$

X=1

X=5

X=11

Answer a question! http://etc.ch/ekXi

GAC-Based Propagation

Pruning the domain of a variable to make a constraint GAC can make a different constraint **no longer GAC**.

Example: $C_1(X, Y) : X > Y$, $C_2(Y, Z) : Y > Z$

$Dom[X] = \{\cancel{1}, \cancel{5}, 11\}, Dom[Y] = \{\cancel{3}, 8, \cancel{15}\}, Dom[Z] = \{4, 6\}$

C2: to make C_2 GAC, we prune 3 from Y

C1: to make C_1 GAC, we prune 5 from X

Need to **re-achieve GAC** for some constraints whenever a domain value is **pruned**.

Answer a question! http://etc.ch/ekXi

GAC: Considerations

- All constraints must be GAC at every node of the search space.
  This is accomplished by **removing** from the domains of the variables all **arc inconsistent values**:
  - Every time we **assign** a value to a variable V, we check all constraints over V and prune **arc inconsistent** values from the **current domain** of the other variables of the constraints.

- Removing a value from a variable domain may trigger **further inconsistency**.
  We have to **repeat** the procedure until **everything is consistent**:
  - Have a queue of constraints that need to be made **GAC**.
  - Constraints are added (back) to the queue if the domain of one of their variables is changed.
  - The procedure **stops** when the queue is empty.

book keeping

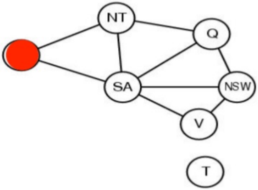
- After backtracking from the current assignment the values that were **pruned** (as a result of that assignment) must be **restored**.
  Some **bookkeeping** needs to be done to remember which values were pruned by which assignment.

$C_1(SA, WA) : SA \neq WA$, $C_2(NT, WA) : NT \neq WA$, $C_3(SA, NT) : SA \neq NT$
$C_4(SA, Q) : SA \neq Q$, $C_5(SA, NSW) : SA \neq NSW$, $C_6(SA, V) : SA \neq V$
$C_7(NT, Q) : NT \neq Q$, $C_8(Q, NSW) : Q \neq NSW$, $C_9(NSW, V) : NSW \neq V$

Value Assignments: $WA = R$
Then, for SA and NT, R becomes arc inconsistent wrt C_1 and C_2.

Current Domains:

$Dom[SA] = \{\cancel{R}, G, B\}$ $Dom[NT] = \{\cancel{R}, G, B\}$
$Dom[Q] = \{R, G, B\}$ $Dom[NSW] = \{R, G, B\}$
$Dom[V] = \{R, G, B\}$ $Dom[T] = \{R, G, B\}$



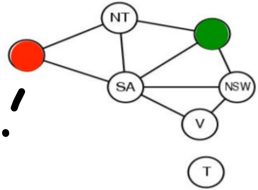
$C_1(SA, WA) : SA \neq WA$, $C_2(NT, WA) : NT \neq WA$, $C_3(SA, NT) : SA \neq NT$
$C_4(SA, Q) : SA \neq Q$, $C_5(SA, NSW) : SA \neq NSW$, $C_6(SA, V) : SA \neq V$
$C_7(NT, Q) : NT \neq Q$, $C_8(Q, NSW) : Q \neq NSW$, $C_9(NSW, V) : NSW \neq V$

Value Assignments: $WA = R, Q = G$
Then, for SA, NT and NSW, G becomes arc inconsistent wrt C_4, C_7, and C_8.

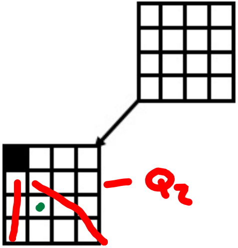
Current Domains:

$Dom[SA] = \{\cancel{G}, \cancel{B}\}$ $Dom[NT] = \{\cancel{G}, \cancel{B}\}$
$Dom[Q] = \{\cancel{R}, G, \cancel{B}\}$ $Dom[NSW] = \{R, \cancel{G}, B\}$
$Dom[V] = \{R, G, B\}$ $Dom[T] = \{R, G, B\}$

DWO!



Answer a question! http://etc.ch/ekXi



Value Assignments: $Q_1 = 1$
Then $Q_2 = 1, Q_2 = 2, Q_3 = 1, Q_3 = 3, Q_4 = 1, Q_4 = 4$
become arc inconsistent.

$C(Q_1,Q_2)$ $C(Q_1,Q_3)$ $C(Q_1,Q_4)$
$C_4(Q_2,Q_3)$ $C_5(Q_2,Q_4)$ $C_6(Q_3,Q_4)$

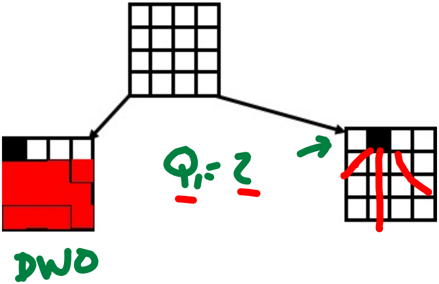
Current Domains:

$Dom[Q_2] = \{\cancel{1}, \cancel{2}, \cancel{3}, 4\}$ $Dom[Q_3] = \{\cancel{1}, 2, \cancel{3}, \cancel{4}\}$ $Dom[Q_4] = \{\cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}\}$

C_4 prune 3 from Dom(Q_2) + 4 from Dom(Q_3)
C_6 results in a DWO for Q_4

Answer a question! http://etc.ch/ekXi

Value Assignments: $Q_1 = 2$
Then $Q_2 = 1, Q_2 = 2, Q_2 = 3, Q_3 = 2,$
$Q_3 = 4, Q_4 = 2$ become arc inconsistent.



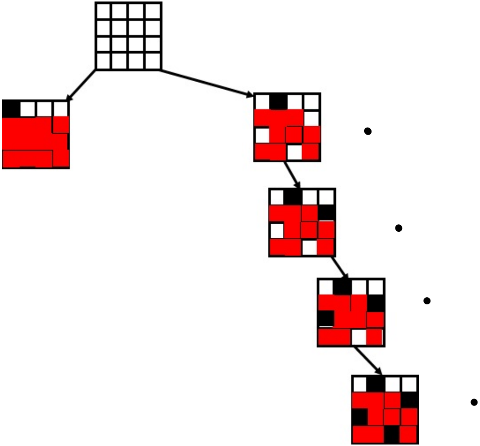
Current Domains:

$Dom[Q_2] = \{\cancel{1}, \cancel{2}, \cancel{3}, 4\}$ $\quad Dom[Q_3] = \{1, \cancel{2}, \cancel{3}, \cancel{4}\}$ $\quad Dom[Q_4] = \{\cancel{1}, \cancel{2}, 3, \cancel{4}\}$

C(Q2 Q3)
C(Q2 Q4)
C(Q3 Q4)

.

Current Domains: $Dom[Q_2] = \{4\}, Dom[Q_3] = \{1\}, Dom[Q_4] = \{3\}$.
Now search no longer has to branch since only one value left for each variable.
It just walks down to a solution assigning each variable in turn.



GAC-Based Propagation

- **Plain Backtracking** check a constraint only when it has **zero** unassinged variables.
- **Forward checking** checks a constraint only when it has **one** unassinged variables.
- **GAC** checks **all constraints**, leading to much more pruning in general.
  - Even at the **root** before any variables have been assigned, we can get **some pruning** by making the constraints GAC consistent.
  - Checking for consistency can be done as a **pre-processing step**, or it can be **directly integrated** into a search algorithm.
  - If we apply arc consistency propagation **during** search the search tree's size will typically be **much reduced** in size.
  - **Note:** GAC enforce **does NOT** find a solution! (why?)
    To find **a solution** we must use do **search** while enforcing GAC.

Answer a question! http://etc.ch/ekXi

$X = \{a_1, a_2\}$

$Y = \{b_1, b_2\}$

$C_1(X, Y)$

	X	Y
T	a_1	b_1
T	a_2	b_2

$C_2(X, Y)$

	X	Y
T	a_1	b_2
T	a_2	b_1

GAC: The Algorithm

```
def GAC_Enforce()
// GAC-Queue contains all constraints one of whose variables has
// had its domain reduced. At the root of the search tree we can
// first run GAC_Enforce with all constraints on GAC-Queue
1.  while GACQueue not empty
2.     C = GACQueue.extract()
3.     for V := each member of scope(C)
4.        for d := CurDom[V]
5.           Find an assignment A for all other variables in scope(C)
             such that C(A ∪ V=d) is True
6.           if A not found
7.              CurDom[V] = CurDom[V] - d   # remove d from the domain of V
8.              if CurDom[V] == {}     # DWO for V
9.                  empty GACQueue
10.                 return DWO        # return immediately
11.           else
12.                 push all constraints C' such that V ∈ scope(C')
                     and C' ∉ GACQueue on to GACQueue
13.  return TRUE   # loop exited without DWO
```

GAC: The Algorithm

```
def GAC(Level)
    if all Variables assigned
        PRINT Value of each Variable
        EXIT or RETURN                  # EXIT for only one solution
                                        # RETURN for more solutions
    V := PickUnassignedVariable()
    Assigned[V] := TRUE
    for d := each member of CurDom(V)
        Value[V] := d
        Prune all values other than d from CurDom[V]
        for each constraint C whose scope contains V
                Put C on GACQueue
        if(GAC_Enforce() != DWO)
                GAC(Level+1) # all constraints were ok
        RestoreAllValuesPrunedByFCCheck()      GAC
     Assigned[V] := FALSE     # UNDO as we have tried all of V's values
     RETURN
```

DFS

GAC: Outcomes

When all constraints are GAC three outcomes are possible:

1. Each domain has a single value.

2. At least one domain is empty.

3. Some domains have more than one value.
   Need to solve this new CSP (usually) **simpler** problem: same constraints, domains have been reduced

- **BT worst-case running time:** $\mathcal{O}(d^N)$, where d is the max size of a variable domain, and N is the number of variables.

- **Worst-case time complexity** of arc consistency procedure on a problem with N variables, c **binary constraints**, and d be the max size of a variable domain:

  - How often will we prune the domain of variable V? $O(d)$

  - How many constraints will be put on the queue when pruning domain of a variable V? $O(\text{degree}(V))$

  - Sum of degrees of all variables: $2 \cdot c$

  - Overall, how many constraints will be put on the queue? $\underline{cd}$

  - Checking consistency of each constraint: d^2

  - **Overall Time Complexity:** $*$ $O(cd^3) < O(d^N)$

More readings:

Bessiere, C., and Regin, J.C. 1997. Arc consistency for general constraint networks: preliminary results. In Proceedings of IJCAI97, 398-404.

GAC: Improving Efficiency

A **support** for a value assignment $V = d$ in a constraint C is an **assignment** A to all of the other variables in $scope(C)$ s.t. $A \cup \{V = d\}$ satisfies C.

A constraint C is **GAC** if for **every** variable V_i in its scope, **every** value $d_i \in CurDomain(V_i)$ has a **support** in C.

C(x y) x=1 C(x y z) (1, 1, 1)

GAC: Improving Efficiency

- Smarter implementations keep track of **supports** to avoid having to search though all possible assignments to the other variables for a satisfying assignment.

- Rather than search for a satisfying assignment to C containing $V = d$, they check if the **current support** is **still valid**.

- Also they take advantage that a support for $V = d$, e.g. $\{V = d, X = a, Y = b, Z = c\}$ is also a support for $X = a, Y = b$, and $Z = c$.

- Another key development in practice is that for some constraints this computation can be done in polynomial time.
  Example: Ideas from graph matching theory are used to find support for variables in $All - diff(V_1, .., V_n)$ in **polynomial time**.

The special purpose algorithms for achieving GAC on particular types of constraints are very important in practice.

(a) $Dom[X] = \{1, 2, 3, 4\}$

(b) $Dom[Y] = \{1, 2, 3, 4\}$

(c) $Dom[Z] = \{1, 2, 3, 4\}$

(d) $Dom[W] = \{1, 2, 3, 4, 5\}$

Do this problem w/ GAC Queue in another order!!

And 3 constraints:

(a) $C_1(X, Y, Z)$ which is satisfied only when $X = Y + Z$

(b) $C_2(X, W)$ which is satisfied only when $W > X$

(c) $C_3(X, Y, Z, W)$ which is satisfied only when $W = X + Z + Y$

~~C1~~ C3

~~C2~~ C2

~~C3~~ C1

~~C1~~

~~C2~~

C1: $X = \{2, 3, 4\}$ $Y = Z = \{1, 2, 3\}$

C2: $W = \{3, 4, 5\}$

C3: $W = \{4, 5\}$ $X = \{2, 3\}$ $Y, Z = \{1, 2\}$

C1: no prunes

C2: " "

- C1(V1,V2,V3)

V1	V2	V3
A	B	C
B	A	C
A	A	B

- C2(V1,V3,V4,V5)

V1	V3	V4	V5
A	A	A	A
A	B	C	B
B	C	B	B
C	A	B	C
C	B	A	B

- C3(V2,V3,V5)

V2	V3	V5
A	A	A
A	B	C
B	C	B
C	A	B
C	B	A

- Dom[V1]…Dom[V5] = {a, b, c}