Information Theory and Linear Regression

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Information Theory

How do we choose between splits when constructing decision trees?

- Measure how much information we can gain from a given split.
- This quantity is call *Information Gain*!
- It is an information theoretic concept that quantifies for a r.v. how much uncertainty is removed if we know its value.

Let's review some information theory basics and definitions.

Uncertainty and Entropy

Uncertainty is like the main building block of many information theory concepts.

- We don't always have all the information about all the variables we care about.
- We use probabilities about events to make *informed* guesses.
- As we learn more information, we can increase confidence, or decrease uncertainty, in our guess.

Uncertainty and Entropy

- Uncertainty is the main building block of many information theory concepts.
- This uncertainty is quantified as Entropy of the random variable, $H(X)$. Mathematically,

For a discrete r.v.:

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x)$$

For a continuous r.v.:

$$H(X) = -\int_{\mathcal{X}} p(x) \log_2 p(x) dx$$

Joint Entropy

- We might be interested in the uncertainty in two or more r.v.s that have some joint distribution.
- This is quantified as the Joint Entropy of the r.v.s in question.
- Its mathematical definition follows analogously to that of entropy but with joint probabilities.

$$H(X,Y) = -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log_2 p(x,y)$$

Exercise: Can you write down the continuous version of this definition?

Conditional Entropy

- We are often interested in the uncertainty in one r.v. once we know the value of another.
- This is quantified as the Conditional Entropy of the first *given* the second.
- Its mathematical definition follows analogously to that of entropy with conditional probabilities.

$$H(Y|X) = -\sum_{x \in \mathcal{X}} p(x) H(Y|X = x)$$

Conditional Entropy

We can expand the terms further:

$$\begin{aligned} H(Y|X) &= -\sum_{x\in\mathcal{X}} p(x)H(Y|X=x) \\ &= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x)p(y|x)\log_2 p(y|x) \\ &= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log_2 p(y|x) \end{aligned}$$

Exercise: Continuous version?

Aside: Logarithm Properties

Some useful properties of logs

- $\log(ab) = \log a + \log b$
- $\log(a/b) = \log a - \log b$

For instance, in the previous slide we encountered $\log_2 p(y|x)$ which can be written as

$$\log_2 \frac{p(x,y)}{p(x)} = \log_2 p(x,y) - \log_2 p(x)$$

Information Gain

Finally, we can now quantify a notion of Information Gain, aka Mutual Information between r.v.s X and Y.

- This quantifies how much more certain (or less uncertain) we are about Y if we know the value of X.
- In other words, how much uncertainty (or entropy) is reduced in Y once we are *given* X?
- Definition: take the entropy of Y and subtract the conditional entropy of Y given X.

$$IG(Y|X) = H(Y) - H(Y|X)$$

Exercises: Information Theory

We now practice computing some of these quantities and prove some standard equalities and inequalities of information theory, which appear in many contexts in machine learning and elsewhere.

Exercise 1

Let $p(x, y)$ be given by

	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Compute

- $H(X), H(Y)$
- $H(X|Y), H(Y|X)$
- $H(X, Y)$
- $IG(Y|X)$

Exercise 2

Prove that entropy $H(X)$ is non-negative, i.e., $H(X) \geq 0$.
For reference, we can use the discrete definition:

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x)$$

Exercise 3

Prove the Chain Rule for entropy, i.e.

$$H(X,Y) = H(X|Y) + H(Y) = H(Y|X) + H(X)$$

Exercise 4

Prove that $H(X, Y) \geq H(X)$.
Hint: you can use results of the first two exercises.

Linear Regression Review

Linear Regression is the problem of predicting a target variable y as a linear combination of input features $\mathbf{x}$.

Fixed inputs given to us:

- Features: $\mathbf{x} = (x_1, x_2, \ldots, x_D) \in \mathbb{R}^D$
- Targets: $t \in \mathbb{R}$

Parameters that we initialize and learn:

- Weights: $\mathbf{w} = (w_1, w_2, \ldots, w_D) \in \mathbb{R}^D$
- Bias: $b \in \mathbb{R}$

Data, Parameters and the Model

- Data is provided to us as $(\mathbf{x}, t)$ tuples.
- Weights and biases, $\mathbf{w}$ and b, are parameters we need to learn.
- We model the predictions y as:

$$\begin{aligned} y = f(\mathbf{x}) &= \sum_{i=1}^{D} w_i x_i + b \\ &= \mathbf{w}^T \mathbf{x} + b \end{aligned}$$

We need to find $\mathbf{w}$ and b such that y is close to the ground truth t.

Objective Function

To learn and evaluate the linear regression model, we need a measure of "closeness", formally called a Loss or Objective Function, which we need to minimize.

- Squared Error Loss: $\mathcal{L}(y, t) = \frac{1}{2}(y - t)^2$.
- For N data samples, we average the individual losses over all samples:

$$
\begin{aligned}
\mathcal{J}(\mathbf{w}) &= \frac{1}{2N} \sum_{i=1}^{N} (y^{(i)} - t^{(i)})^2 \\
&= \frac{1}{2N} \sum_{i=1}^{N} (\mathbf{w}^T \mathbf{x}^{(i)} + b - t^{(i)})^2
\end{aligned}
$$

Exercise: Linear Regression Bias-Variance

Assume the optimal weights are given by $\mathbf{w}^*$ and for all data samples

$$t^{(i)} = \mathbf{w}^{*T}\mathbf{x}^{(i)} + \epsilon^{(i)}$$

where $\epsilon^{(i)}$ are independent random noise variables.
Further, recall that the loss function is given by

$$\mathcal{J}(w) = \frac{1}{2N}||\mathbf{y} - \mathbf{t}||^2$$

Exercise: Linear Regression Bias-Variance

Using the above, derive the bias-variance decomposition for the linear regression problem.