# CSC 311: Introduction to Machine Learning

## Lecture 3 - Bagging, Linear Models I

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University of Toronto, Summer 2023

# Outline

# Announcements

- HW1 is due on June 05 (10% late penalty for each late day, no credit after 3 days).
- I have arranged TA office hours (on website) for the assignment.
- Go to the earliest possible ones you can attend.
- **Manage your time well!** If you wait till the last TA session, you may a long wait to ask your question.

# Today

- Today we will introduce ensembling methods that combine multiple models and can perform better than the individual members.
  - We've seen many individual models (KNN, decision trees)
- We will see bagging:
  - Train models independently on random "resamples" of the training data.
- We will introduce linear regression, our first parametric learning algorithm.
  - This will exemplify how we'll think about learning algorithms for the rest of the course.

# Bias/Variance Decomposition

- prediction $y$ at a query $\mathbf{x}$ is a random variable (where the randomness comes from the choice of dataset),
- $y_\star$ is the optimal deterministic prediction, and
- $t$ is a random target sampled from the true conditional $p(t|\mathbf{x})$.

$$\mathbb{E}[(y-t)^2] = \underbrace{(y_\star - \mathbb{E}[y])^2}_{\text{bias}} + \underbrace{\text{Var}(y)}_{\text{variance}} + \underbrace{\text{Var}(t)}_{\text{Bayes error}}$$

# Bias/Variance Decomposition

$$\mathbb{E}[(y-t)^2] = \underbrace{(y_\star - \mathbb{E}[y])^2}_{\text{bias}} + \underbrace{\text{Var}(y)}_{\text{variance}} + \underbrace{\text{Var}(t)}_{\text{Bayes error}}$$

Bias/variance decomposes the expected loss into three terms:

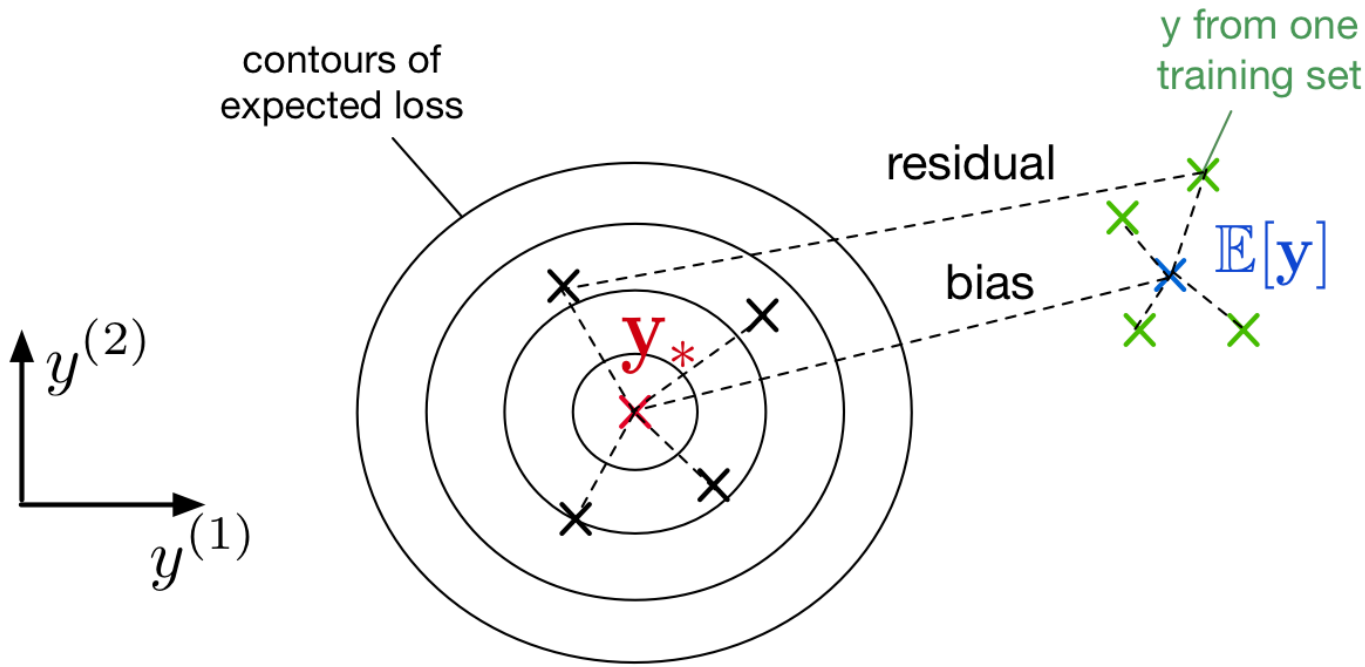
- bias: how wrong the expected prediction is (corresponds to under-fitting)
- variance: the amount of variability in the predictions (corresponds to over-fitting)
- Bayes error: the inherent unpredictability of the targets

Often loosely use "bias" for "under-fitting" and "variance" for "over-fitting".

# Visualizing Bias/Variance Decomposition

An overly **simple** model (e.g. KNN with large $k$) might have

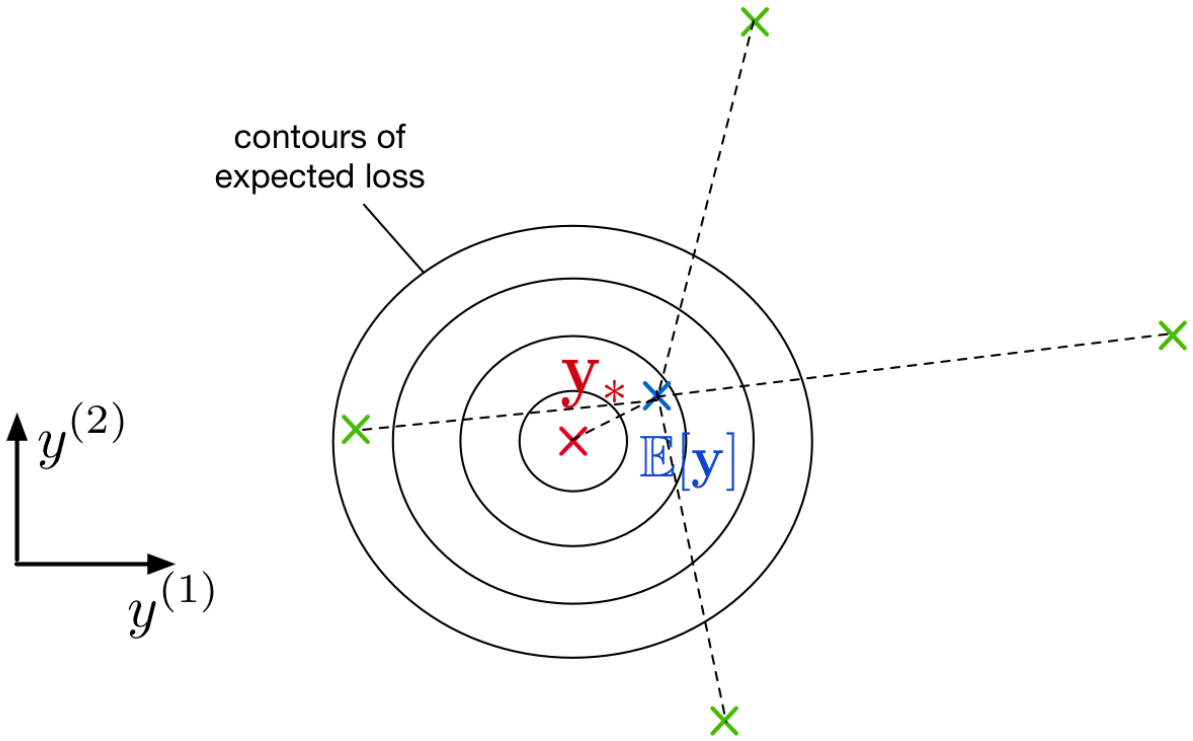
- **high bias**
  (cannot capture the structure in the data)
- **low variance**
  (enough data to get stable estimates)



# Visualizing Bias/Variance Decomposition

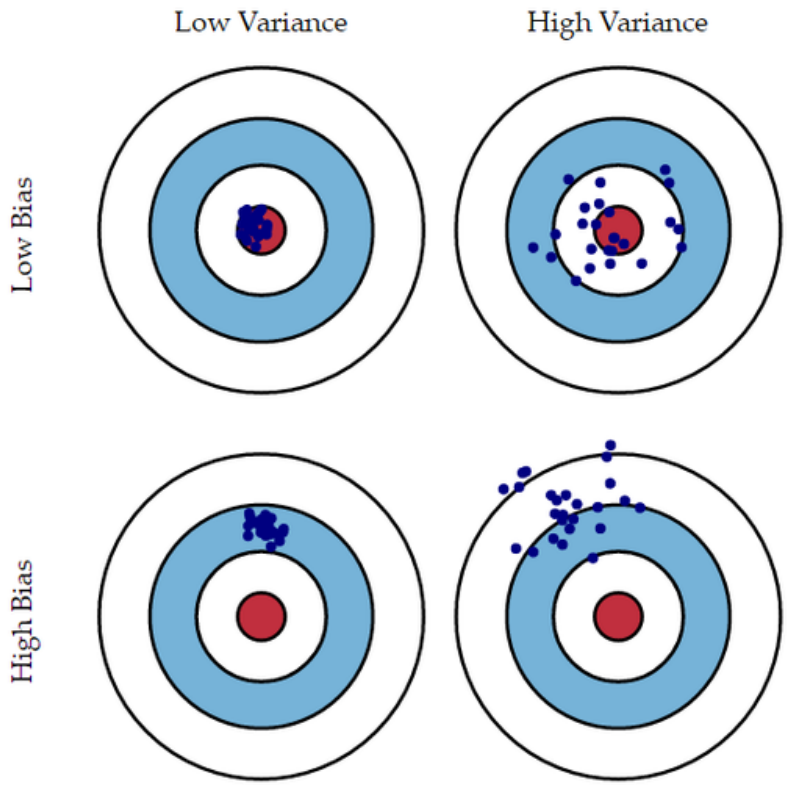
An overly **complex** model (e.g. KNN with $k = 1$) might have

- **low bias**
  ( learns all the relevant structure)
- **high variance**
  (fits the quirks of the data you happened to sample)



# Bias/Variance Decomposition: Another Visualization

- The following graphic summarizes the previous two slides:




A: Bayes error

# Bagging Motivation

- Sample $m$ independent training sets from $p_{\text{sample}}$.
- Compute the prediction $y_i$ using each training set.
- Compute the average prediction $y = \frac{1}{m}\sum_{i=1}^{m} y_i$.
- How does this affect the three terms of the expected loss?
  - **Bias:** unchanged,
    since the averaged prediction has the same expectation

$$\mathbb{E}[y] = \mathbb{E}\left[\frac{1}{m}\sum_{i=1}^{m} y_i\right] = \mathbb{E}[y_i]$$

  - **Variance:** reduced,
    since we are averaging over independent predictions

$$\text{Var}[y] = \text{Var}\left[\frac{1}{m}\sum_{i=1}^{m} y_i\right] = \frac{1}{m^2}\sum_{i=1}^{m}\text{Var}[y_i] = \frac{1}{m}\text{Var}[y_i].$$

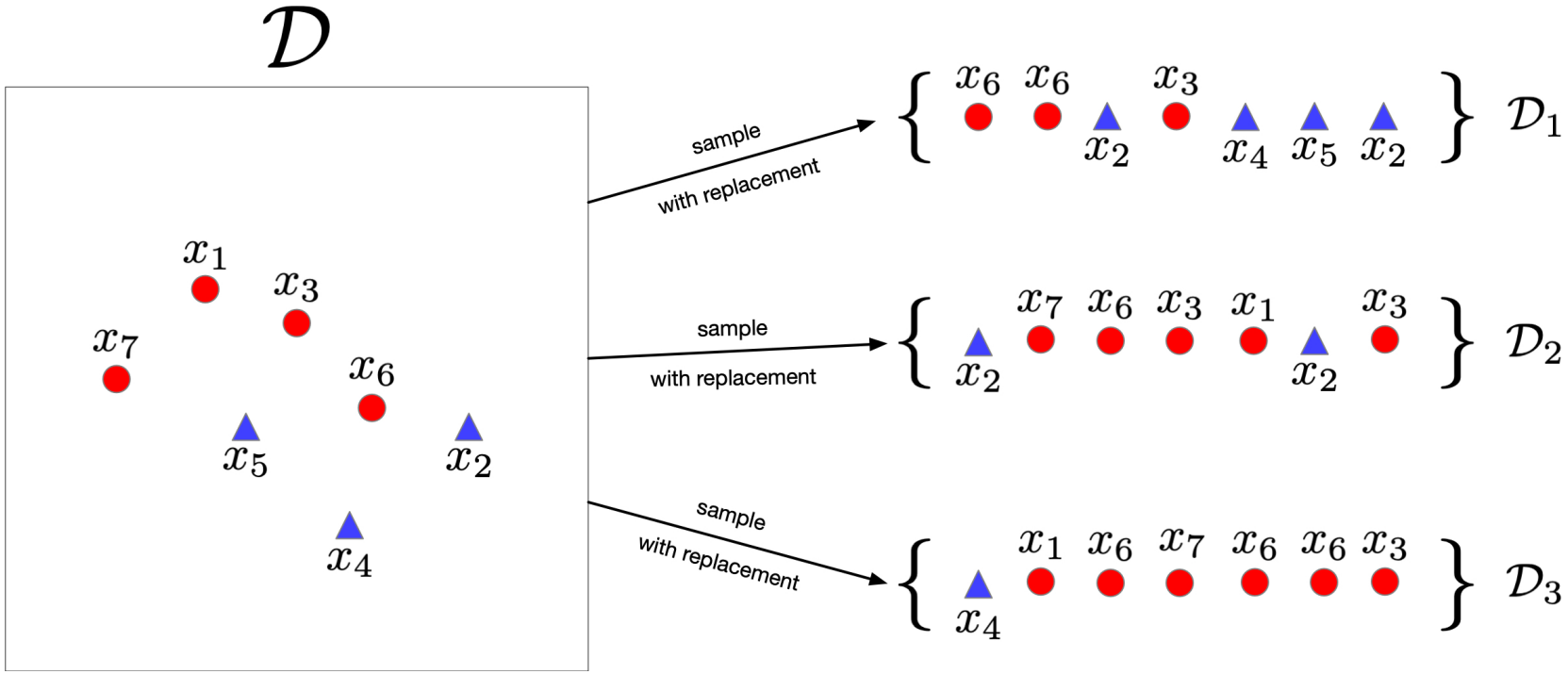
  - **Bayes error:** unchanged,
    since we have no control over it

# Bagging: The Idea

- In practice, $p_{\text{sample}}$ is often expensive to sample from. So training separate models on independently sampled datasets is very wasteful of data!
- Given training set $\mathcal{D}$, use the empirical distribution $p_{\mathcal{D}}$ as a proxy for $p_{\text{sample}}$. This is called bootstrap aggregation or bagging .
  - Take a dataset $\mathcal{D}$ with $n$ examples.
  - Generate $m$ new datasets ("resamples" or "bootstrap samples")
  - Each dataset has $n$ examples sampled from $\mathcal{D}$ with replacement.
  - Average the predictions of models trained on the $m$ datasets.
- One of the most important ideas in statistics!
  - Intuition: As $|\mathcal{D}| \to \infty$, we have $p_{\mathcal{D}} \to p_{\text{sample}}$.

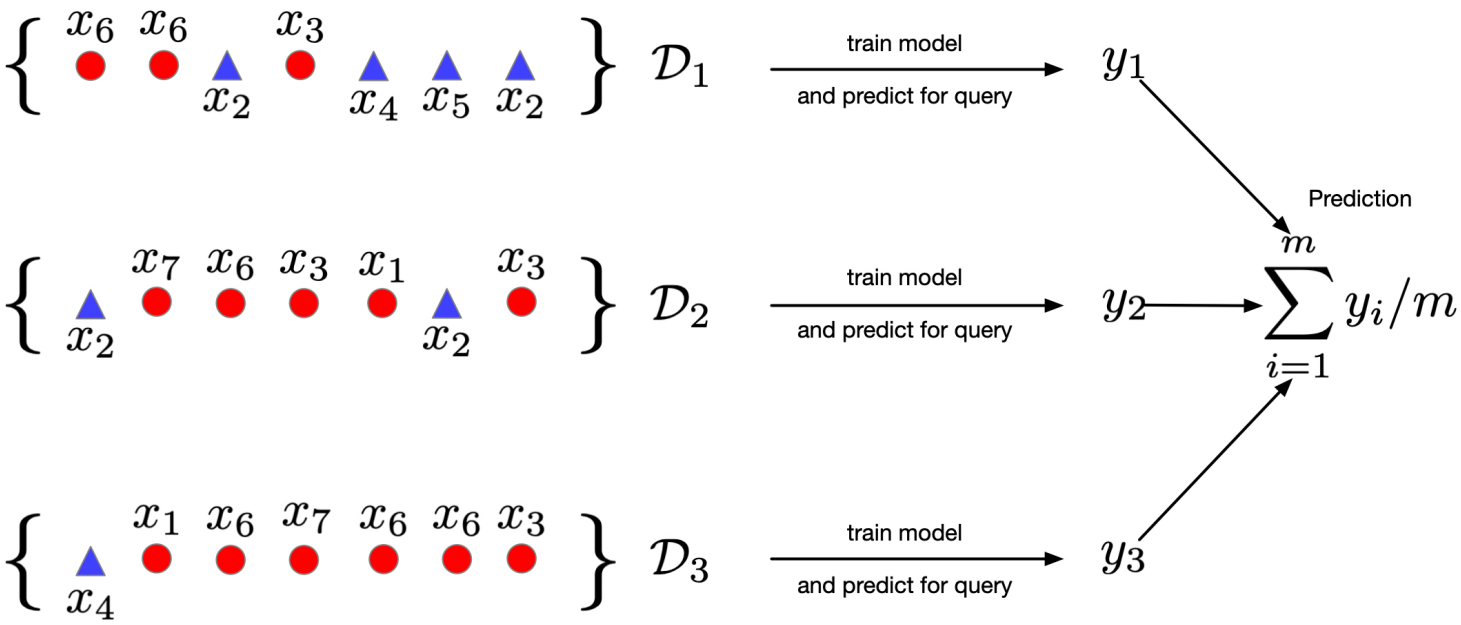
# Bagging Example 1/2

Create $m = 3$ datasets by sampling from $D$ with replacement.
Each dataset contains $n = 7$ examples.



# Bagging Example 2/2

Generate prediction $y_i$ using dataset $D_i$.
Average the predictions.



# Aggregating Predictions for Binary Classification

- Classifier $i$ outputs a prediction $y_i$
- $y_i$ can be real-valued $y_i \in [0, 1]$ or a binary value $y_i \in \{0, 1\}$
- Average the predictions and apply a threshold.

$$y_{\text{bagged}} = \mathbb{I}\left(\frac{1}{m}\sum_{i=1}^{m} y_i > 0.5\right)$$

- Same as majority vote.

# Bagging Properties

- A bagged classifier can be stronger than the average model.
  - E.g. on "Who Wants to be a Millionaire", "Ask the Audience" is much more effective than "Phone a Friend".
- But, if $m$ datasets are NOT independent, don't get the $\frac{1}{m}$ variance reduction.
- Reduce correlation between datasets by introducing *additional* variability
  - Invest in a diversified portfolio, not just one stock.
  - Average over multiple algorithms, or multiple configurations of the same algorithm.

# Random Forests

- A trick to reduce correlation between bagged decision trees:
  For each node, choose a random subset of features
  and consider splits on these features only.
- Probably the best black-box machine learning algorithm.
  - works well with no tuning.
  - widely used in Kaggle competitions.

# Bagging Summary

Reduces over-fitting by averaging predictions.

In most competition winners.
A small ensemble often better than a single great model.

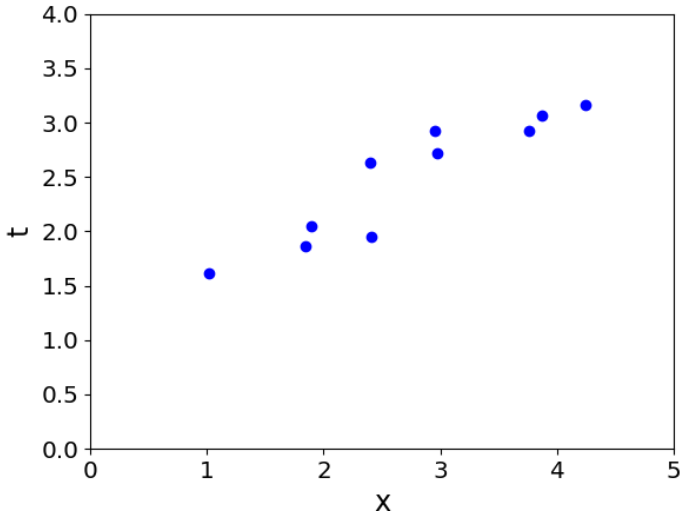
Limitations:

- Does not reduce bias in case of squared error.
- Correlation between classifiers means less variance reduction.
  Add more randomness in Random Forests.
- Weighting members equally may not be the best.
  Weighted ensembling often leads to better results if members are very different.

# Overview

- Third learning algorithm of the course: linear regression.
  - Task: predict scalar-valued targets (e.g. stock prices)
  - Architecture: linear function of the inputs
- While KNN was a complete algorithm, linear regression exemplifies a modular approach that will be used throughout this course:
  - choose a model describing the relationships between variables of interest
  - define a loss function quantifying how bad the fit to the data is
  - choose a regularizer saying how much we prefer different candidate models (or explanations of data)
  - fit a model that minimizes the loss function and satisfies the constraint/penalty imposed by the regularizer, possibly using an optimization algorithm
- Mixing and matching these modular components give us a lot of new ML methods.

# Supervised Learning Setup




In supervised learning:

- There is input $\mathbf{x} \in \mathcal{X}$, typically a vector of features (or covariates)
- There is target $t \in \mathcal{T}$ (also called response, outcome, output, class)
- Objective is to learn a function $f : \mathcal{X} \to \mathcal{T}$ such that $t \approx y = f(\mathbf{x})$ based on some data $\mathcal{D} = \{(\mathbf{x}^{(i)}, t^{(i)}) \text{ for } i = 1, 2, ..., N\}$.

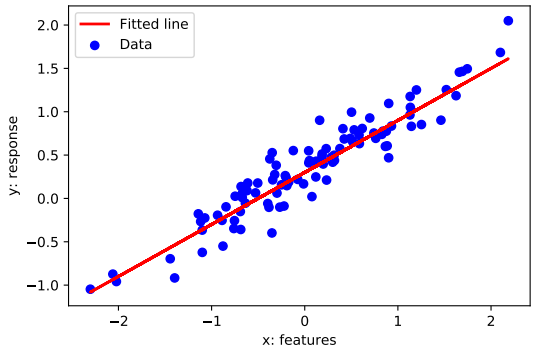
# Linear Regression - Model

- Model: In linear regression, we use a *linear* function of the features $\mathbf{x} = (x_1, \ldots, x_D) \in \mathbb{R}^D$ to make predictions $y$ of the target value $t \in \mathbb{R}$:

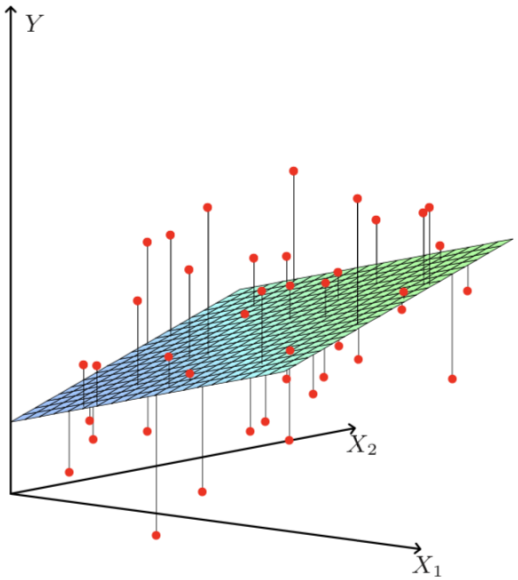
$$y = f(\mathbf{x}) = \sum_j w_j x_j + b$$

  - $y$ is the prediction
  - $\mathbf{w}$ is the weights
  - $b$ is the bias (or intercept)
- $\mathbf{w}$ and $b$ together are the parameters
- We hope that our prediction is close to the target: $y \approx t$.

# What is Linear? 1 feature vs D features




- If we have only 1 feature:
  $y = wx + b$ where $w, x, b \in \mathbb{R}$.
- $y$ is linear in $x$.




- If we have $D$ features:
  $y = \mathbf{w}^\top \mathbf{x} + b$ where $\mathbf{w}, \mathbf{x} \in \mathbb{R}^D$,
  $b \in \mathbb{R}$
- $y$ is linear in $\mathbf{x}$.

Relation between the prediction $y$ and inputs $\mathbf{x}$ is linear in both cases.

# Linear Regression - Loss Function

- A loss function $\mathcal{L}(y, t)$ defines how bad it is if, for some example $\mathbf{x}$, the algorithm predicts $y$, but the target is actually $t$.
- Squared error loss function:

$$\mathcal{L}(y, t) = \frac{1}{2}(y - t)^2$$

- $y - t$ is the residual, and we want to make this small in magnitude
- The $\frac{1}{2}$ factor is just to make the calculations convenient.
- Cost function: loss function averaged over all training examples

$$\begin{aligned}\mathcal{J}(\mathbf{w}, b) &= \frac{1}{2N}\sum_{i=1}^{N}\left(y^{(i)} - t^{(i)}\right)^2 \\ &= \frac{1}{2N}\sum_{i=1}^{N}\left(\mathbf{w}^\top \mathbf{x}^{(i)} + b - t^{(i)}\right)^2\end{aligned}$$

- Terminology varies. Some call "cost" *empirical* or *average loss*.

# Vectorization

- The prediction for one data point can be computed using a for loop:

```
y = b
for j in range(M):
    y += w[j] * x[j]
```

- Excessive super/sub scripts are hard to work with, and Python loops are slow, so we vectorize algorithms by expressing them in terms of vectors and matrices.

$$\mathbf{w} = (w_1, \ldots, w_D)^\top \qquad \mathbf{x} = (x_1, \ldots, x_D)^\top$$

$$y = \mathbf{w}^\top \mathbf{x} + b$$

- This is simpler and executes much faster:

```
y = np.dot(w, x) + b
```

# Vectorization

Why vectorize?

- The equations, and the code, will be simpler and more readable. Gets rid of dummy variables/indices!
- Vectorized code is much faster
  - Cut down on Python interpreter overhead
  - Use highly optimized linear algebra libraries (hardware support)
  - Matrix multiplication very fast on GPU (Graphics Processing Unit)

Switching in and out of vectorized form is a skill you gain with practice

- Some derivations are easier to do element-wise
- Some algorithms are easier to write/understand using for-loops and vectorize later for performance

# Vectorization

- We can organize all the training examples into a design matrix $\mathbf{X}$ with one row per training example, and all the targets into the target vector $\mathbf{t}$.

one feature across all training examples

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}^{(1)\top} \\ \mathbf{x}^{(2)\top} \\ \mathbf{x}^{(3)\top} \end{pmatrix} = \begin{pmatrix} 8 & 0 & 3 & 0 \\ 6 & -1 & 5 & 3 \\ 2 & 5 & -2 & 8 \end{pmatrix}$$

one training example (vector)

- Computing the predictions for the whole dataset:

$$\mathbf{X}\mathbf{w} + b\mathbf{1} = \begin{pmatrix} \mathbf{w}^T\mathbf{x}^{(1)} + b \\ \vdots \\ \mathbf{w}^T\mathbf{x}^{(N)} + b \end{pmatrix} = \begin{pmatrix} y^{(1)} \\ \vdots \\ y^{(N)} \end{pmatrix} = \mathbf{y}$$

# Vectorization

- Computing the squared error cost across the whole dataset:

$$\mathbf{y} = \mathbf{X}\mathbf{w} + b\mathbf{1}$$

$$\mathcal{J} = \frac{1}{2N}\|\mathbf{y} - \mathbf{t}\|^2$$

- Sometimes we may use $\mathcal{J} = \frac{1}{2}\|\mathbf{y} - \mathbf{t}\|^2$, without a normalizer. This would correspond to the sum of losses, and not the averaged loss. The minimizer does not depend on $N$ (but optimization might!).
- We can also add a column of 1's to design matrix, combine the bias and the weights, and conveniently write

$$\mathbf{X} = \begin{bmatrix} 1 & [\mathbf{x}^{(1)}]^\top \\ 1 & [\mathbf{x}^{(2)}]^\top \\ 1 & \vdots \end{bmatrix} \in \mathbb{R}^{N \times (D+1)} \quad \text{and } \mathbf{w} = \begin{bmatrix} b \\ w_1 \\ w_2 \\ \vdots \end{bmatrix} \in \mathbb{R}^{D+1}$$

Then, our predictions reduce to $\mathbf{y} = \mathbf{X}\mathbf{w}$.

# Solving the Minimization Problem

Our goal is to minimize the cost function $\mathcal{J}(\mathbf{w})$.

Recall from calculus: the minimum of a smooth function (if it exists) occurs at a critical point, i.e. point where the derivative is zero.

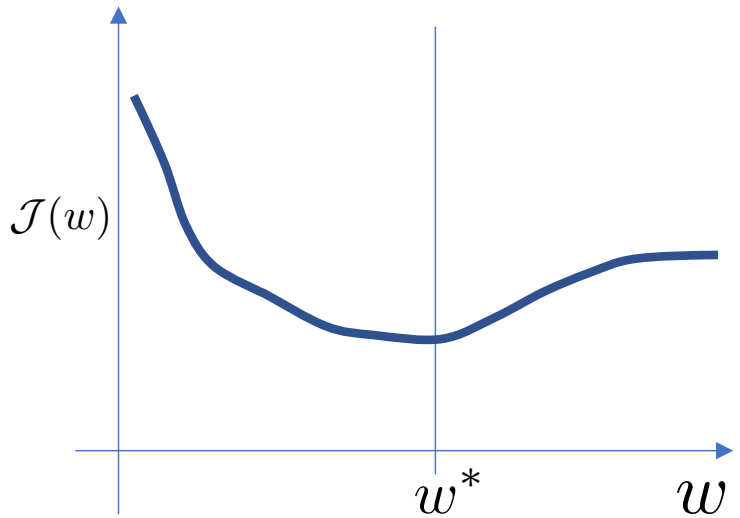
$$\nabla_{\mathbf{w}} \mathcal{J} = \frac{\partial \mathcal{J}}{\partial \mathbf{w}} = \begin{pmatrix} \frac{\partial \mathcal{J}}{\partial w_1} \\ \vdots \\ \frac{\partial \mathcal{J}}{\partial w_D} \end{pmatrix}$$

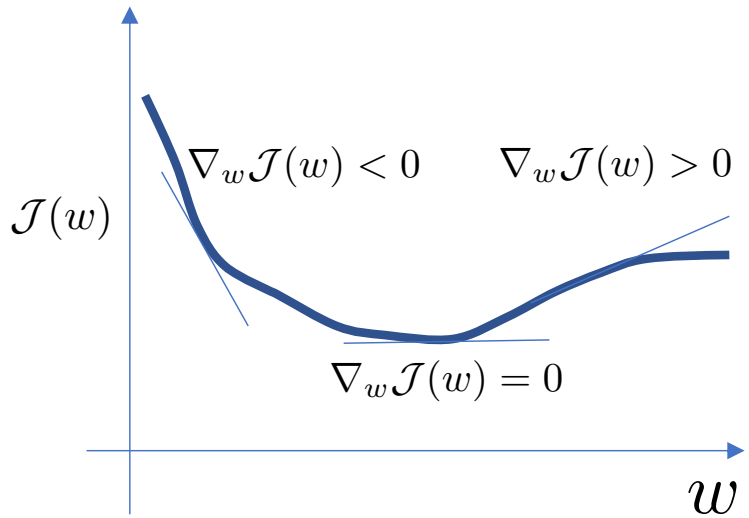
Solutions may be direct or iterative.

- Direct solution: set the gradient to zero and solve in closed form — directly find provably optimal parameters.
- Iterative solution: repeatedly apply an update rule that gradually takes us closer to the solution.

# Direct Solution: Calculus

- Lets consider a cartoon visualization of $\mathcal{J}(w)$ where $w$ is single dimensional
- **Left** We seek $w = w^*$ that minimizes $\mathcal{J}(w)$
- **Right** The gradients of a function can tell us where the maxima and minima of functions lie
- **Strategy:** Write down an algebraic expression for $\nabla_w \mathcal{J}(w)$. Set equation to 0. Solve for $w$



# Direct Solution: Calculus

- We seek $\mathbf{w}$ to minimize $\mathcal{J}(\mathbf{w}) = \frac{1}{2}\|\mathbf{X}\mathbf{w} - \mathbf{t}\|^2$
- Taking the gradient with respect to $\mathbf{w}$ and setting it to $\mathbf{0}$, we get:

$$\nabla_{\mathbf{w}} \mathcal{J}(\mathbf{w}) = \mathbf{X}^\top \mathbf{X}\mathbf{w} - \mathbf{X}^\top \mathbf{t} = \mathbf{0}$$

  See course notes for additional details.
- Optimal weights:

$$\mathbf{w}^* = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{t}$$

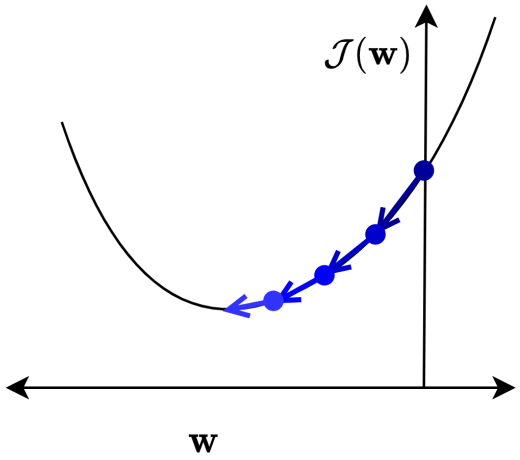
- Linear regression is one of only a handful of models in this course that permit direct solution.

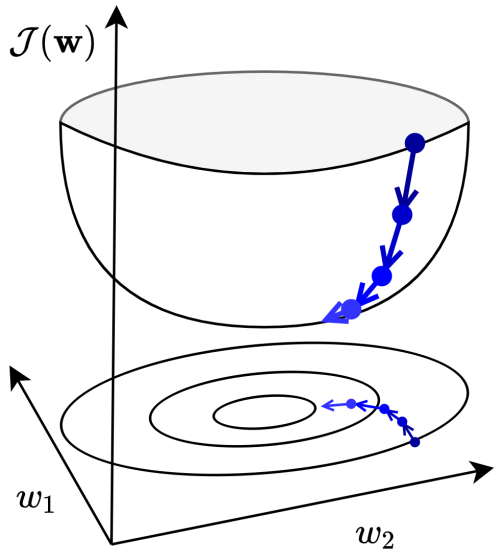
# Iterative solution: Gradient Descent

- Many optimization problems don't have a direct solution.
- A more broadly applicable way to minimize the cost function: gradient descent.
- Gradient descent is an iterative algorithm, which means we apply an update repeatedly until some criterion is met.
- We initialize the weights to something reasonable (e.g. all zeros) and repeatedly adjust them in the direction of steepest descent.

# Gradient Descent

- Observe:
  - if $\partial \mathcal{J}/\partial w_j > 0$, then decreasing $\mathcal{J}$ requires decreasing $w_j$.
  - if $\partial \mathcal{J}/\partial w_j < 0$, then decreasing $\mathcal{J}$ requires increasing $w_j$.




- The following update always decreases the cost function for small enough $\alpha$ (unless $\partial \mathcal{J}/\partial w_j = 0$):

$$w_j \leftarrow w_j - \alpha \frac{\partial \mathcal{J}}{\partial w_j}$$

# Gradient Descent

- The following update always decreases the cost function for small enough $\alpha$ (unless $\partial \mathcal{J}/\partial w_j = 0$):

$$w_j \leftarrow w_j - \alpha \frac{\partial \mathcal{J}}{\partial w_j}$$

- $\alpha > 0$ is a learning rate (or step size).
  - The larger $\alpha$ is, the faster $\mathbf{w}$ changes.
  - Values are typically small, e.g. 0.01 or 0.0001.
  - We'll see later how to tune the learning rate.
  - If cost is the total loss rather than average loss, a smaller learning rate will be needed ($\alpha' = \alpha/N$).

# Gradient Descent

- Gradient descent gets its name from the gradient.
  Recall the definition of the gradient:

$$\nabla_{\mathbf{w}} \mathcal{J} = \frac{\partial \mathcal{J}}{\partial \mathbf{w}} = \begin{pmatrix} \frac{\partial \mathcal{J}}{\partial w_1} \\ \vdots \\ \frac{\partial \mathcal{J}}{\partial w_D} \end{pmatrix}$$

  The gradient is the direction of fastest *increase* in $\mathcal{J}$.
- Update rule in vector form:

$$\mathbf{w} \leftarrow \mathbf{w} - \alpha \frac{\partial \mathcal{J}}{\partial \mathbf{w}}$$

- Update rule for linear regression:

$$\mathbf{w} \leftarrow \mathbf{w} - \frac{\alpha}{N} \sum_{i=1}^{N} (y^{(i)} - t^{(i)}) \, \mathbf{x}^{(i)}$$

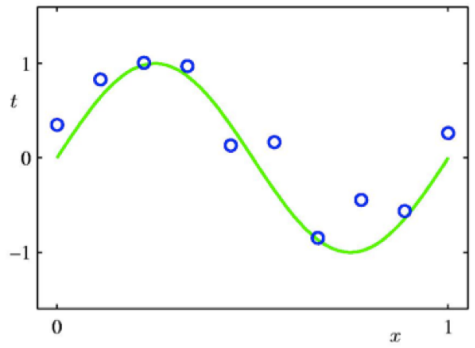
- Gradient descent updates $\mathbf{w}$ in the direction of fastest *decrease*.
- Once it converges, we get a critical point, i.e. $\frac{\partial \mathcal{J}}{\partial \mathbf{w}} = \mathbf{0}$.

# Why Use Gradient Descent?

- GD is applicable to a much broader set of models
- GD is easier to implement than direct solutions
- For regression in high-dimensional space, GD is more efficient than direct solution
  - For example, the linear regression direction solution $(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{t}$ requires matrix inversion, which is $\mathcal{O}(D^3)$.
  - Each GD update costs $\mathcal{O}(ND)$ or less with stochastic gradient descent.
  - Huge difference if $D \gg 1$

# Feature Mapping (Basis Expansion)

Can we use linear regression to model a non-linear relationship?



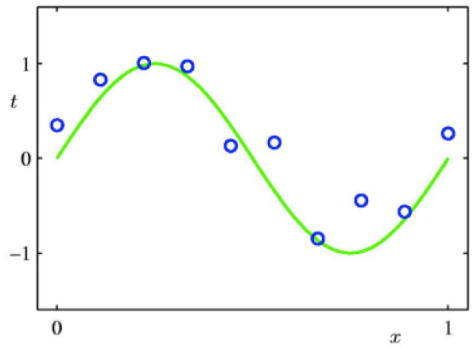

- Map the input features to another space using $\boldsymbol{\psi}(\mathbf{x}) : \mathbb{R}^D \to \mathbb{R}^d$.
- Treat the mapped feature (in $\mathbb{R}^d$) as the input of a linear regression procedure.

# Polynomial Feature Mapping

Fit the data using a degree-$M$ polynomial function of the form:

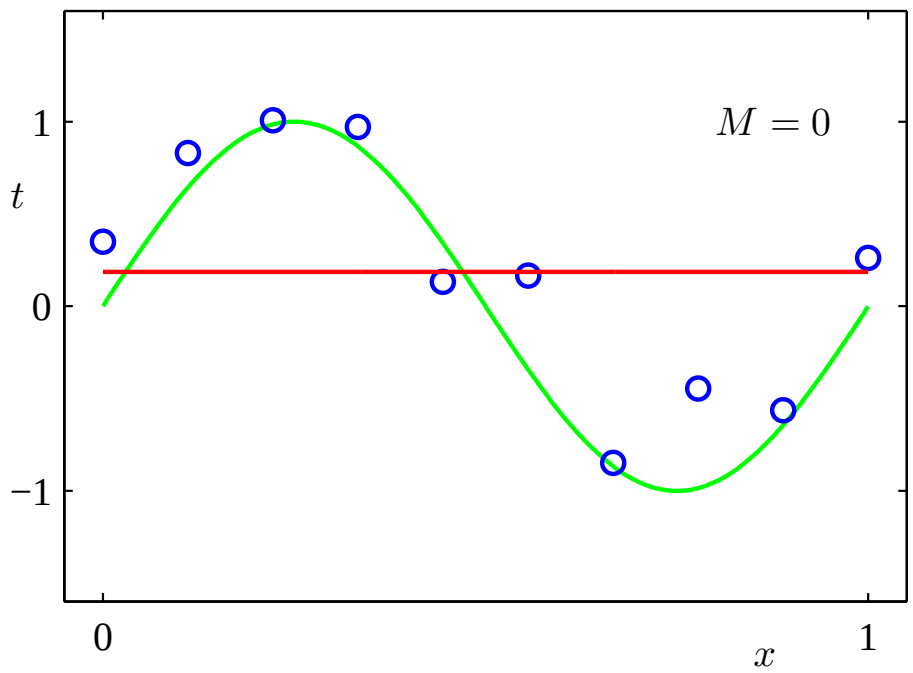
$$y = w_0 + w_1 x + w_2 x^2 + ... + w_M x^M = \sum_{i=0}^{M} w_i x^i$$

- The feature mapping is $\boldsymbol{\psi}(x) = [1, x, x^2, ..., x^M]^\top$.
- $y = \boldsymbol{\psi}(x)^\top \mathbf{w}$ is linear in $w_0, w_1, ....$
- Use linear regression to find $\mathbf{w}$.
- In general, $\boldsymbol{\psi}$ can be any function.
  Another example: $\boldsymbol{\psi}(x) = [1, \sin(2\pi x), \cos(2\pi x), \sin(4\pi x), ...]^\top$.



# Polynomial Feature Mapping with $M = 0$

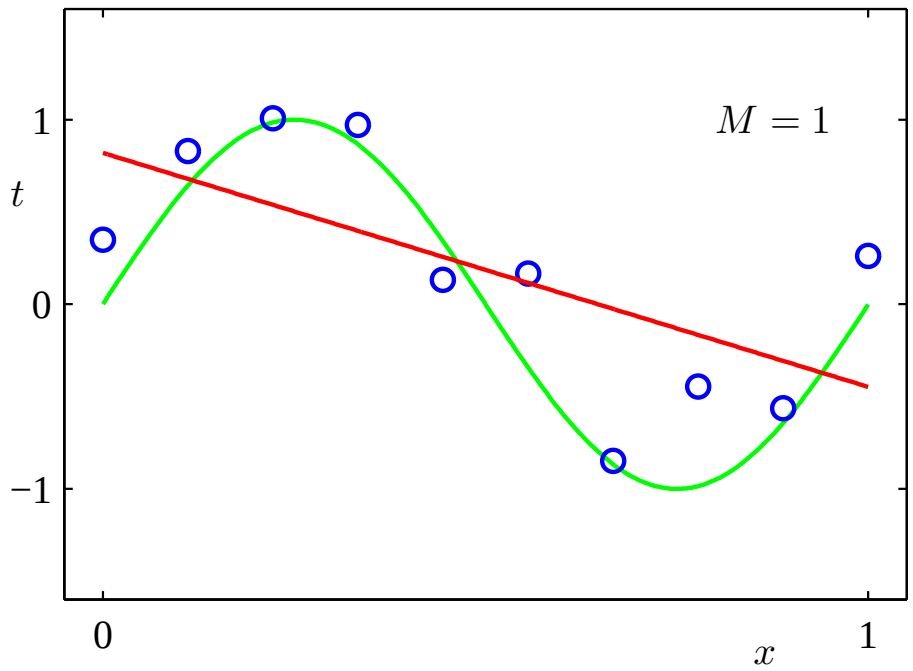
$$y = w_0$$




-Pattern Recognition and Machine Learning, Christopher Bishop.

# Polynomial Feature Mapping with $M = 1$

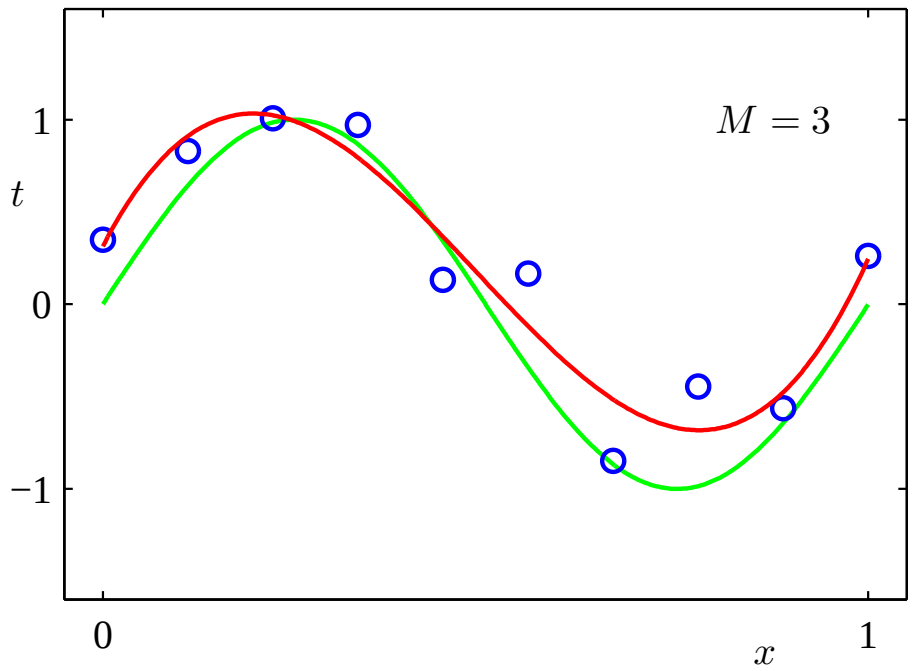
$$y = w_0 + w_1 x$$




-Pattern Recognition and Machine Learning, Christopher Bishop.

# Polynomial Feature Mapping with $M = 3$

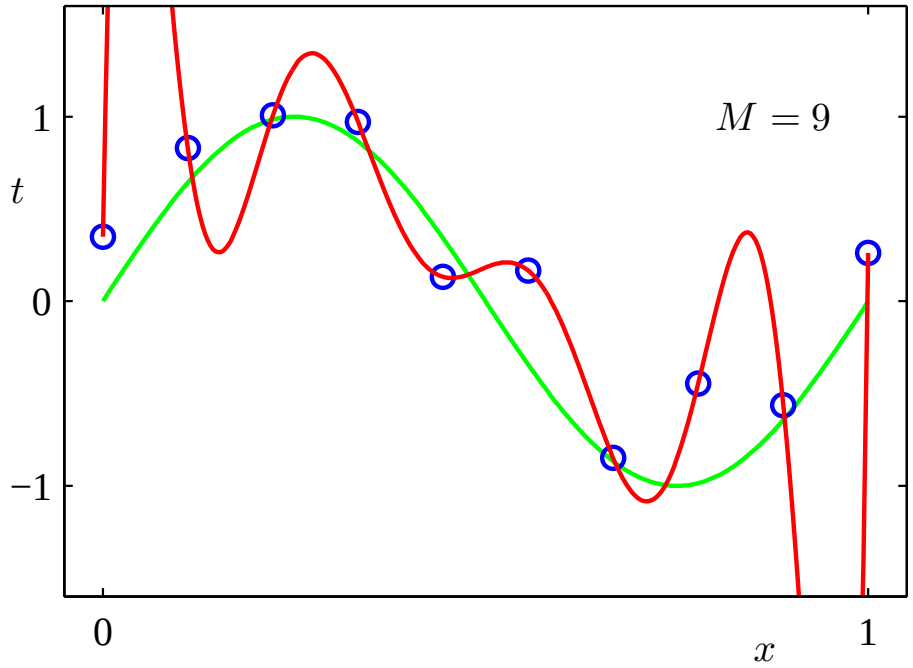
$$y = w_0 + w_1 x + w_2 x^2 + w_3 x^3$$




-Pattern Recognition and Machine Learning, Christopher Bishop.

# Polynomial Feature Mapping with $M = 9$

$$y = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + \ldots + w_9 x^9$$

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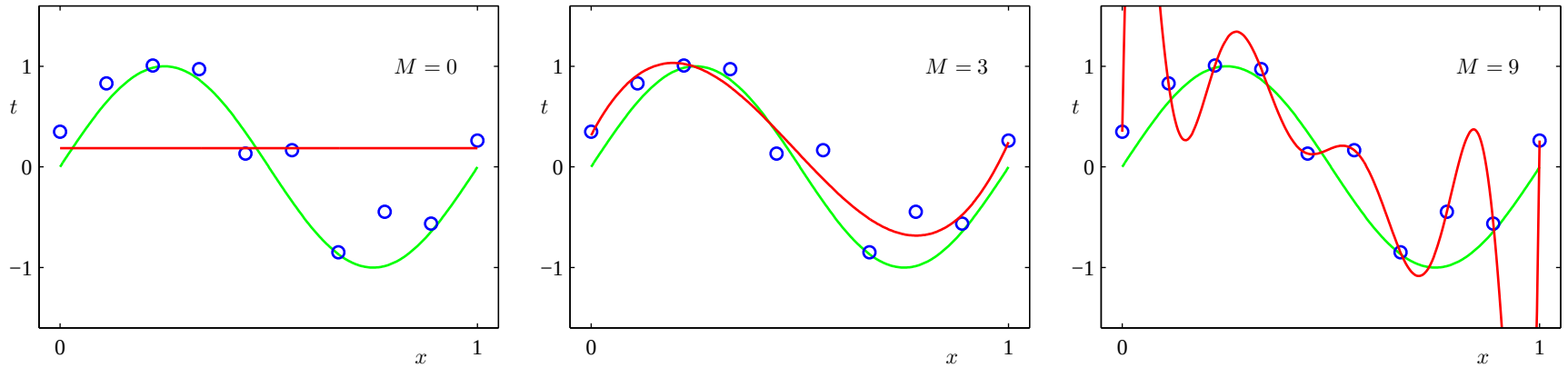
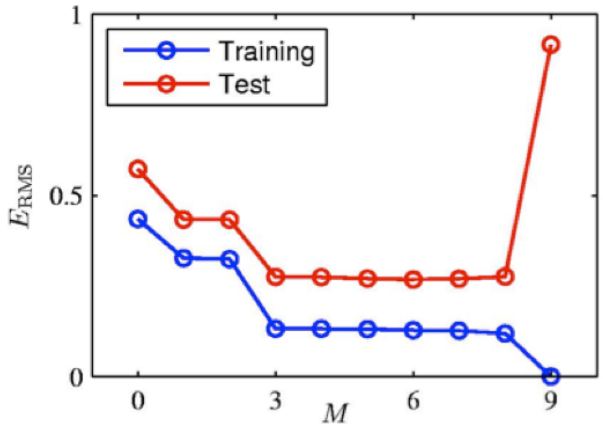
# Model Complexity and Generalization

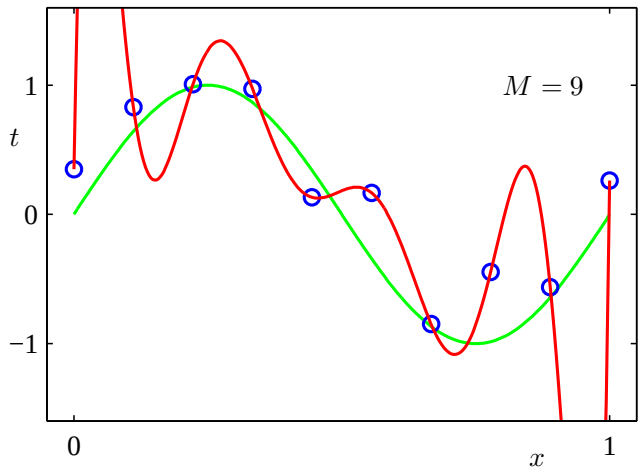



Under-fitting (M=0): Model is too simple, does not fit the data.
Good model (M=3): Achieves small test error, generalizes well.
Over-fitting (M=9): Model is too complex, fits perfectly.



# Model Complexity and Generalization

| | $M=0$ | $M=1$ | $M=3$ | $M=9$ |
|---|---|---|---|---|
| $w_0^\star$ | 0.19 | 0.82 | 0.31 | 0.35 |
| $w_1^\star$ | | -1.27 | 7.99 | 232.37 |
| $w_2^\star$ | | | -25.43 | -5321.83 |
| $w_3^\star$ | | | 17.37 | 48568.31 |
| $w_4^\star$ | | | | -231639.30 |
| $w_5^\star$ | | | | 640042.26 |
| $w_6^\star$ | | | | -1061800.52 |
| $w_7^\star$ | | | | 1042400.18 |
| $w_8^\star$ | | | | -557682.99 |
| $w_9^\star$ | | | | 125201.43 |




- As $M$ increases, the magnitude of coefficients gets larger.
- For $M = 9$, the coefficients have become finely tuned to the data.
- Between data points, the function exhibits large oscillations.

# Regularization

- The degree $M$ of the polynomial controls the model's complexity.
- The value of $M$ is a hyperparameter for polynomial expansion, just like $k$ in KNN. We can tune it using a validation set.
- Restricting the number of parameters / basis functions ($M$) is a crude approach to controlling the model complexity.
- Another approach: keep the model large, but regularize it
  - Regularizer: a function that quantifies how much we prefer one hypothesis vs. another

# $L^2$ (or $\ell_2$ ) Regularization

- Encourage the weights to be small by choosing the $L^2$ penalty as our regularizer.

$$\mathcal{R}(\mathbf{w}) = \tfrac{1}{2}\|\mathbf{w}\|_2^2 = \frac{1}{2}\sum_j w_j^2.$$

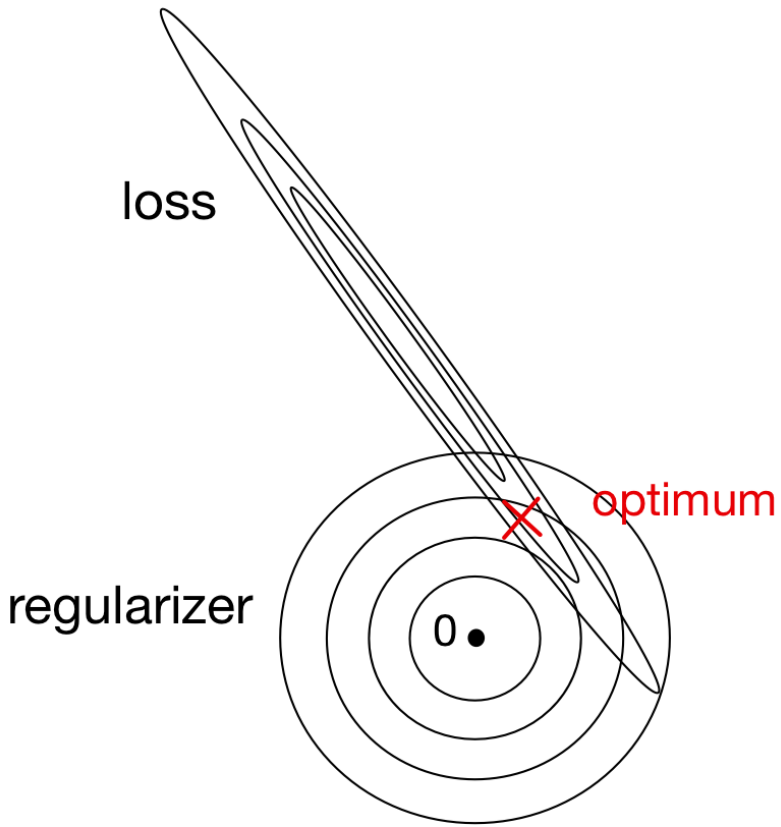
- The regularized cost function makes a tradeoff between the fit to the data and the norm of the weights.

$$\mathcal{J}_{\text{reg}}(\mathbf{w}) = \mathcal{J}(\mathbf{w}) + \lambda\mathcal{R}(\mathbf{w}) = \mathcal{J}(\mathbf{w}) + \frac{\lambda}{2}\sum_j w_j^2$$

- If you fit training data poorly, $\mathcal{J}$ is large.
  If the weights are large in magnitude, $\mathcal{R}$ is large.
- Large $\lambda$ penalizes weight values more.
- $\lambda$ is a hyperparameter we can tune with a validation set.

# $L^2$ (or $\ell_2$ ) Regularization

- The geometric picture:



# $L^2$ Regularized Least Squares: Ridge regression

For the least squares problem, we have $\mathcal{J}(\mathbf{w}) = \frac{1}{2N}\|\mathbf{X}\mathbf{w} - \mathbf{t}\|^2$.

- When $\lambda > 0$ (with regularization), regularized cost gives

$$\begin{aligned}\mathbf{w}_\lambda^{\text{Ridge}} = \operatorname*{argmin}_{\mathbf{w}} \mathcal{J}_{\text{reg}}(\mathbf{w}) &= \operatorname*{argmin}_{\mathbf{w}} \frac{1}{2N}\|\mathbf{X}\mathbf{w} - \mathbf{t}\|_2^2 + \frac{\lambda}{2}\|\mathbf{w}\|_2^2 \\ &= (\mathbf{X}^\top\mathbf{X} + \lambda N\mathbf{I})^{-1}\mathbf{X}^\top\mathbf{t}\end{aligned}$$

- The case $\lambda = 0$ (no regularization) reduces to least squares solution!
- Can also formulate the problem as

$$\operatorname*{argmin}_{\mathbf{w}} \frac{1}{2}\|\mathbf{X}\mathbf{w} - \mathbf{t}\|_2^2 + \frac{\lambda}{2}\|\mathbf{w}\|_2^2$$

with solution

$$\mathbf{w}_\lambda^{\text{Ridge}} = (\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^\top\mathbf{t}.$$

# Gradient Descent under the $L^2$ Regularization

- Gradient descent update to minimize $\mathcal{J}$:

$$\mathbf{w} \leftarrow \mathbf{w} - \alpha \frac{\partial}{\partial \mathbf{w}} \mathcal{J}$$

- The gradient descent update to minimize the $L^2$ regularized cost $\mathcal{J} + \lambda \mathcal{R}$ results in weight decay:

$$\begin{aligned}
\mathbf{w} &\leftarrow \mathbf{w} - \alpha \frac{\partial}{\partial \mathbf{w}} (\mathcal{J} + \lambda \mathcal{R}) \\
&= \mathbf{w} - \alpha \left( \frac{\partial \mathcal{J}}{\partial \mathbf{w}} + \lambda \frac{\partial \mathcal{R}}{\partial \mathbf{w}} \right) \\
&= \mathbf{w} - \alpha \left( \frac{\partial \mathcal{J}}{\partial \mathbf{w}} + \lambda \mathbf{w} \right) \\
&= (1 - \alpha\lambda)\mathbf{w} - \alpha \frac{\partial \mathcal{J}}{\partial \mathbf{w}}
\end{aligned}$$

# Conclusion so far

Linear regression exemplifies recurring themes of this course:

- choose a model and a loss function
- formulate an optimization problem
- solve the minimization problem using one of two strategies
  - direct solution (set derivatives to zero)
  - gradient descent
- vectorize the algorithm, i.e. represent in terms of linear algebra
- make a linear model more powerful using features
- improve the generalization by adding a regularizer