Computer Science
UNIVERSITY OF TORONTO

CSC200, Lecture 22: Recent Extensions to Stable Matching Problems

Guest Lecturer:

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Stable Matching: Big Real World Problem!

- Many large real-world markets which use stable matching literature & results:
  - Residency Matching:
    - NRMP 2015: 34,905 residents
    - Psychology Market: ~ 3100 psychologists
  - School Choice
    - Boston, NYC

Outline

- Background/Refresher
- Stable Matching & Information Burden
  - Partial Preference Information
  - Preference Elicitation
- Stable Matching Problem with Couples (i.e., “two-body problem”)
  - Properties and Solutions
  - Strategic Considerations

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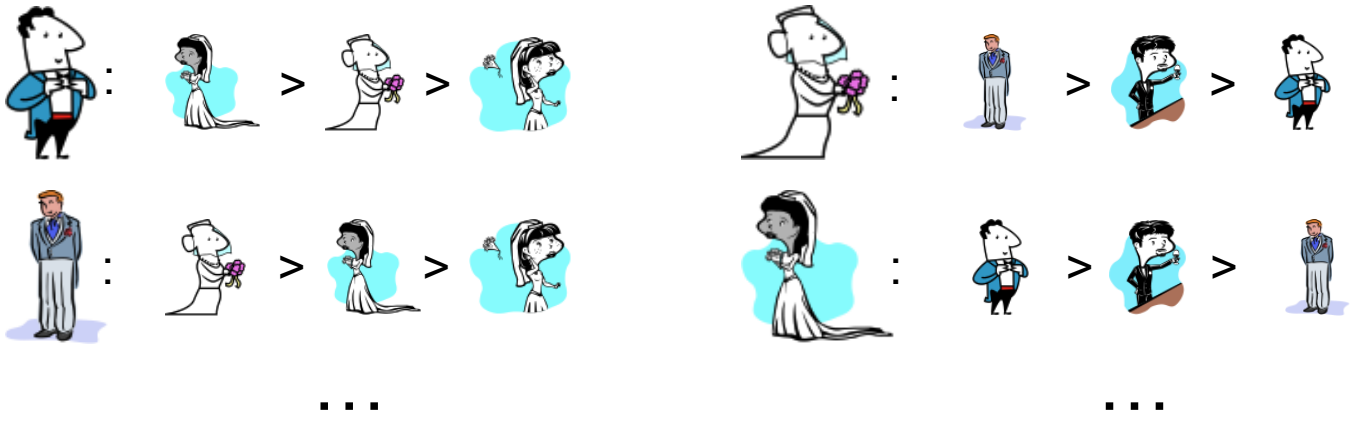
Stable Matching Problem

- Men & women have preferences



- Assign everyone such that no two people prefer each other to their assigned match

Stable Matching Problem

- Notation:
  - Set of women W, set of men M
  - Matching function μ
  - $\varnothing$ denotes a man/woman is "unmatched":
    - $\mu(w) = \varnothing$ (resp. $\mu(m) = \varnothing$) denotes that *w* (resp. *m)* is unmatched
  - $M^+ = M \cup \{\varnothing\}$ (resp. W^+)
- A m*atching* is a mapping μ s.t.
  - Any man *m* is matched to $\mu(m) \in W^+$
  - Any woman *w* is matched to $\mu(w) \in M^+$
  - $\mu(w) = m$ iff $\mu(m) = w$

Stable Matching Problem

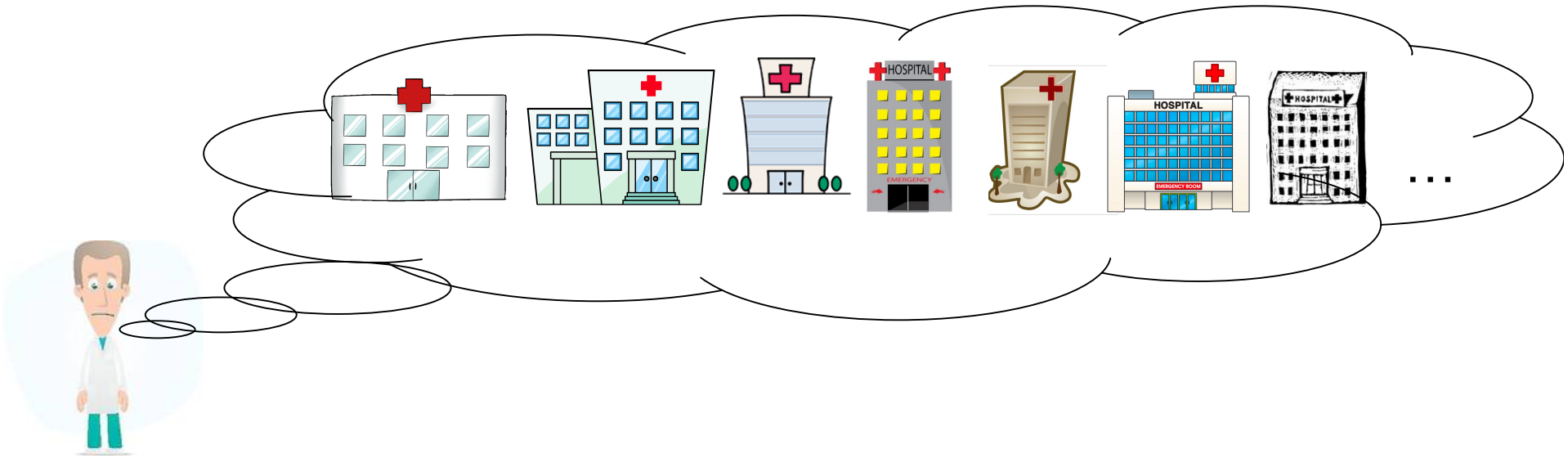
- A matching μ is *stable* iff:
  - μ is *Individually Rational*:
    - If *m* prefers being unmatched than being matched to *w*, $\mu(m) \neq w$
    - Only needed for "incomplete lists"
      - Some alternatives are undesirable
  - No *blocking pairs*:
    - (*m*,*w*) is a blocking pair if:
      - *m* prefers *w* to $\mu(m)$ AND
      - *w* prefers *m* to $\mu(w)$

Stable Matching Problem

- Nice properties:
  - Stable matching guaranteed to exist
  - Stable matching can be found in polynomial time *(O(n^2))*
  - Female-optimal matching exists (and can be found in polynomial time)
    - Resp. male-optimal matching exists
  - Exists mechanism that guarantees women OR men will tell the truth
    - No mechanism exists where both tell the truth

Problems

- Information burden!
  - Thousands of hospitals in residency matching



- Unable to richly express preferences
  - Two people looking for jobs want jobs in the same city
  - Hospitals want a urologist and a cardiologist, but not two urologists

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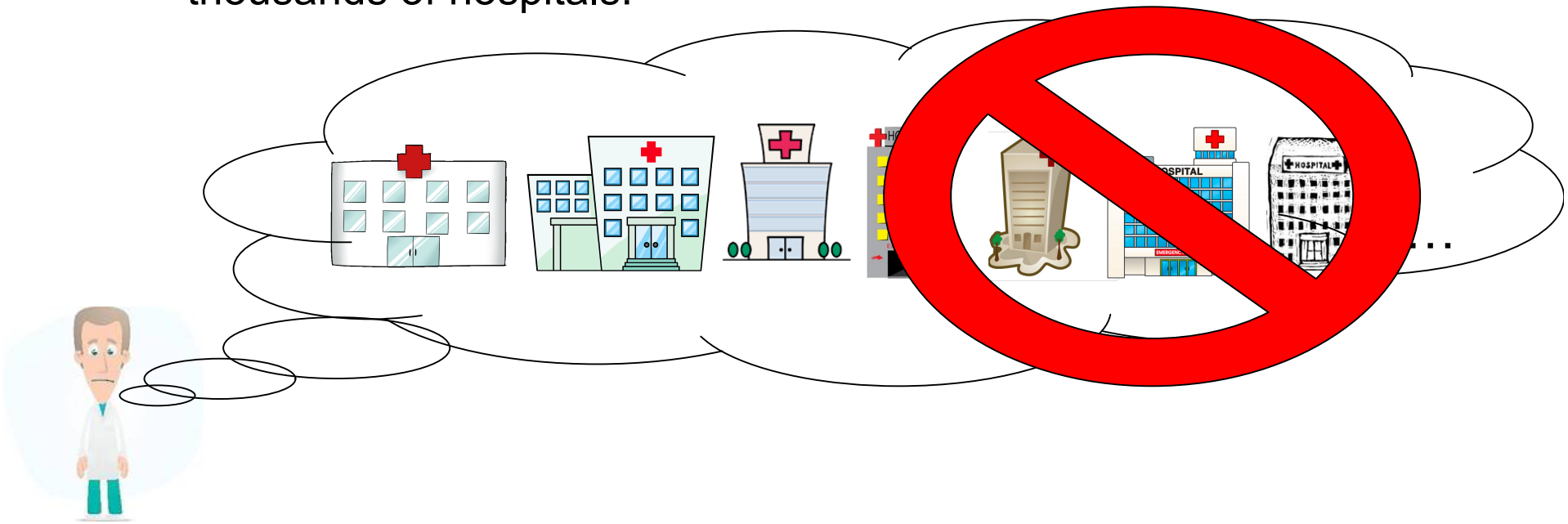
Information Burden in SMP: Preference Bottleneck

- Residency matching: Thousands of new doctors and thousands of hospitals!

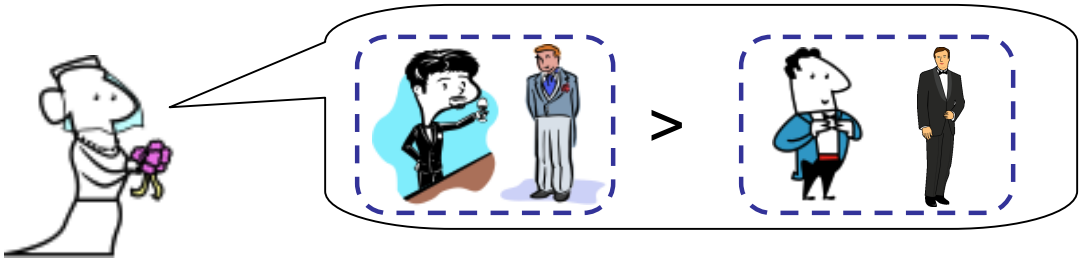


- Very little of this information is needed in practice

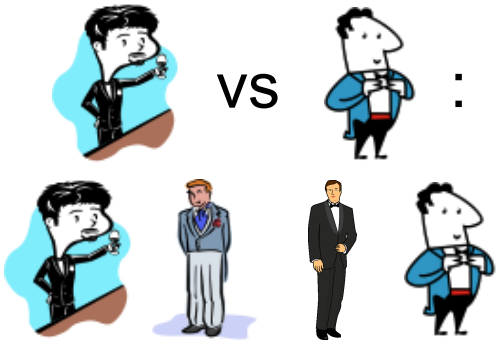
Many different ways to address this

- Examples:
  - Pilotto, E.; Rossi, F.; Venable, K. B.; and Walsh, T. 2009. Compact preference representation in stable marriage problems. In Algorithmic Decision Theory. Springer. 390−401
  - Rastegari, B.; Condon, A.; Immorlica, N.; and Leyton- Brown, K. 2013. Two-sided matching with partial infor- mation. In Proceedings of the Fourteenth ACM Conference on Electronic Commerce, 733−750. ACM.
  - Robin S. Lee and Michael Schwarz. Interviewing in two-sided matching markets. Working Paper 14922, National Bureau of Economic Research, April 2009.
- We're focusing on preference *elicitation*
  - Interactively ask questions to find a guaranteed stable match
    - Drummond, J. and Boutilier, C., 2013, August. Elicitation and approximately stable matching with partial preferences. In Proc. IJCAI-13 (pp. 97-105). AAAI Press.

Matching with Partial Preferences



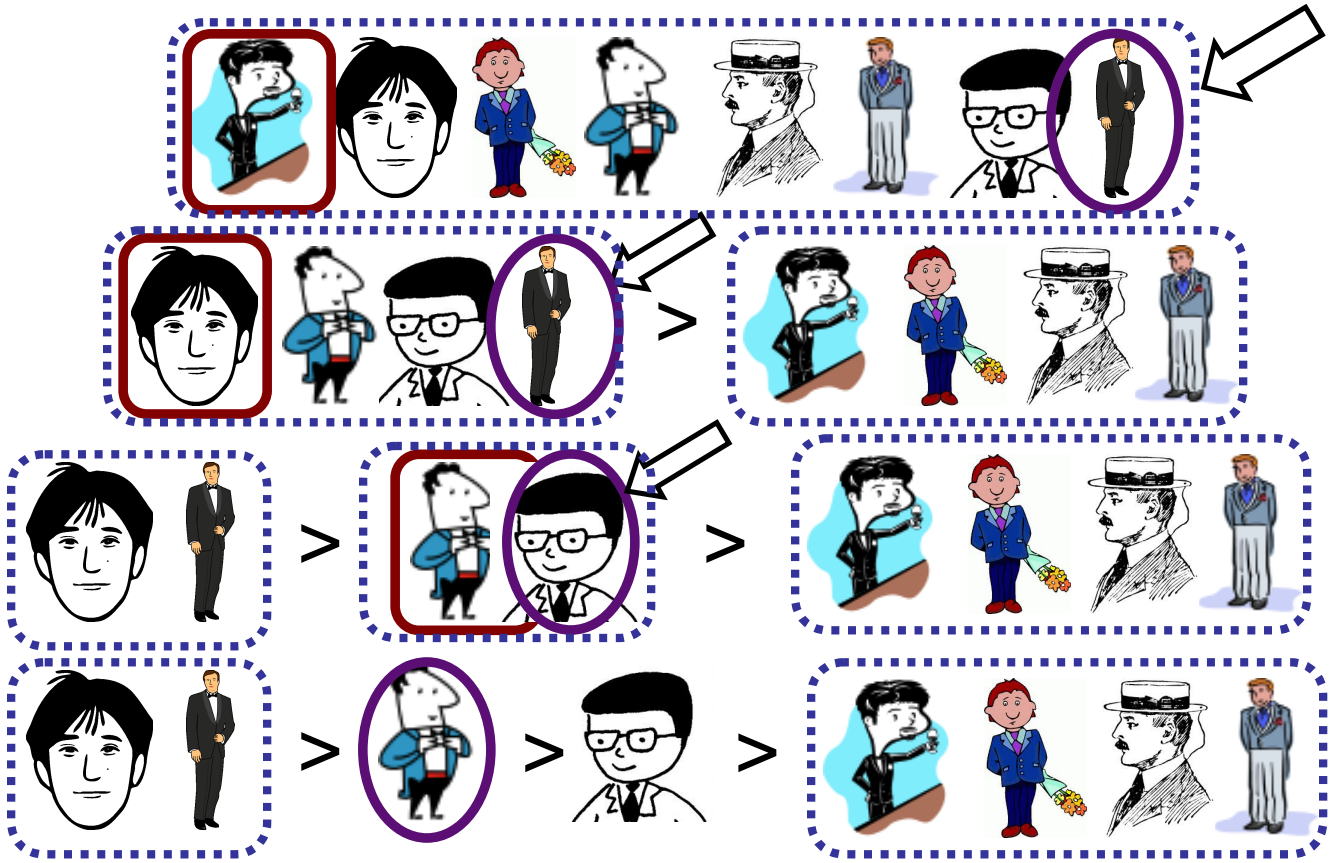
- Limited knowledge about agents' preferences
- How can we match without full information?
  - Look at *worst case*



Partial Information: Worst Case Preferences

- If μ is stable under *worst case completion* of partial preferences, μ is guaranteed stable under *any* completion
- Elicitation Idea. Repeat until μ stable in worst case:
  - Pick a matching μ
  - Identify blocking pair (m,w) in μ with the "most to gain" by defecting in the worst case
  - Ask m and w to give us more information about their preferences

Preference Elicitation: Regret-Based Halving



Until Guaranteed Stable:

Identify worst case blocking pairs *(m,w)*

Ask *m* and *w* to split their appropriate partitions

Discussion

- Works like a targeted binary search over agents' preferences
  - In empirical simulations, $\log_2(n)$ queries
- Potential problems:
  - Still potentially too much information
    - Though less than algorithms like Deferred Acceptance!
  - Lots of information we're not using
    - Attributes ("I prefer to live in big cities.")

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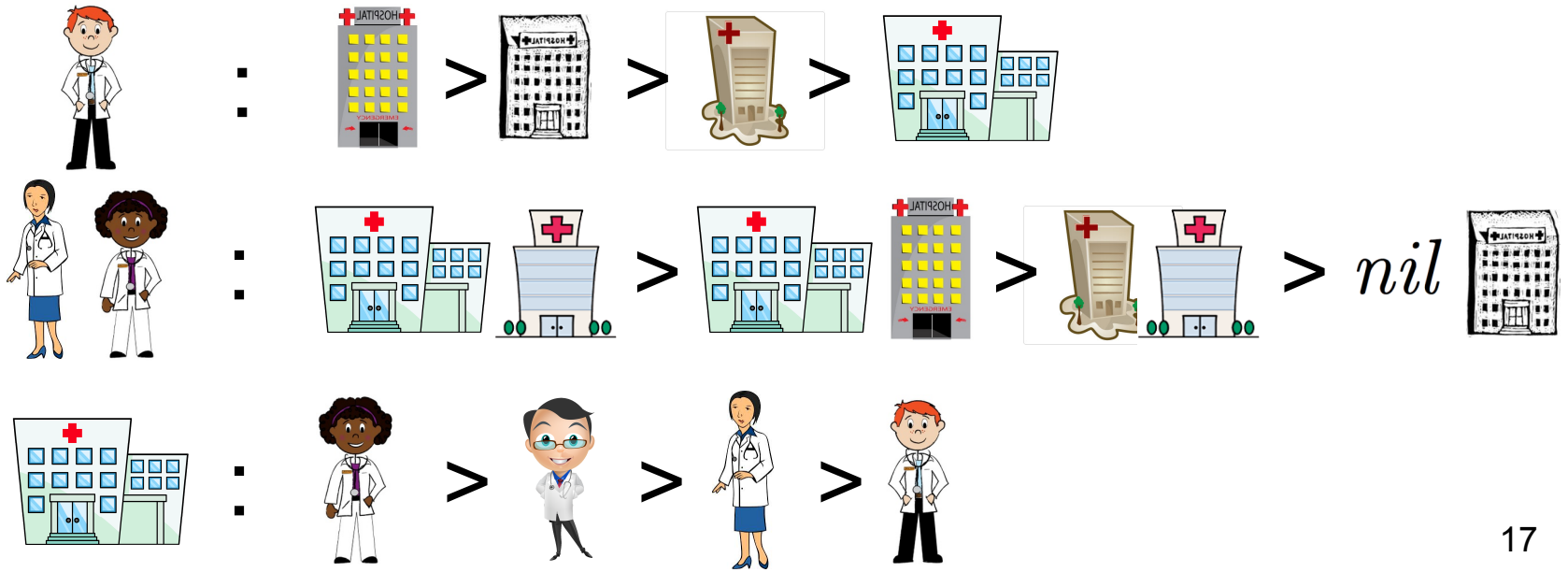
Stable Matching w/Couples (SMP-C)

- Extension of standard SMP
- Match so that no doctor/hospital pair prefers each other to assigned match (extend stability def'n as expected)



Properties of SMP-C

- NP-Complete
  - SMP is $O(n^2)$
- Stable matching not guaranteed to exist
- Important real-world problem
  - NRMP 2015: 34,905 residents, 6% in couples

Traditional Solution Techniques for SMP-C

- Heuristic extensions to canonical Deferred Acceptance (Gale-Shapley)
  - First proposed 1999
  - Not guaranteed to find a solution, even if one exists
  - Cannot verify no solution exists
  - No longer guaranteed to be hospital or resident optimal
  - Not truth-telling

SAT as a new SMP-C Solver

- *SAT is an Effective and Complete Method for Solving Stable Matching Problems with Couples*; **J. Drummond**, A. Perrault, F. Bacchus; Proc. IJCAI-15; 2015
- Main goal: provide a new method for solving SMP-C that is complementary to current methods
  - And addresses current methods' weaknesses

Other Recent SMP-C Solvers

- Develop an Integer Program (IP) Encoding
  - (branch & bound-based optimization)
  - Péter Biró, David F Manlove, and Iain McBride. The hospitals/residents problem with couples: Complexity and integer programming models. In Experimental Algorithms, pages 10–21. Springer, 2014.

What is SAT?

- "Boolean Satisfiability Problem"
- One of the first & most widely studied NP-Complete problems
  - As proved by UofT's Steven Cook in 1971
- Binary variables
- Conjunction of disjunctive clauses
  - e.g., $(x_1 \vee x_2) \wedge (x_3 \vee x_1) \wedge (\sim x_1 \vee x_4) \wedge \ldots$

Why do we care about SAT?

- NP-Complete: *any* NP-complete problem can be rewritten as SAT
  - Including SMP-C!!
- Widely studied: many out-of-the-box SAT solvers
  - Work well in practice on a variety of problems
  - Huge advances in the last decade
- SAT is a natural representation for *configuration* problems

SMP-C as a SAT Encoding

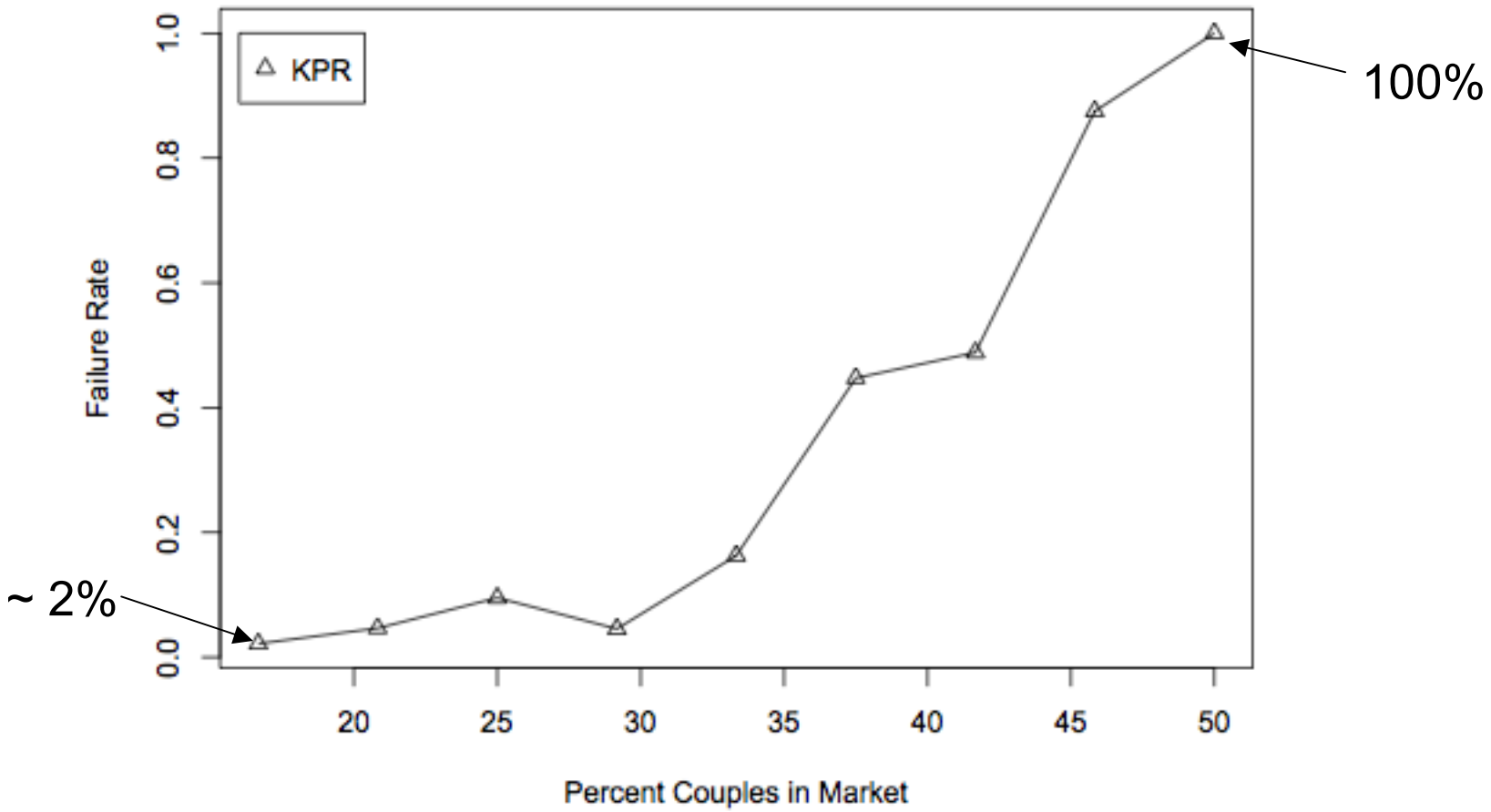
- Change the SMP-C problem into a SAT problem so that any solution to the SAT encoding provides a solution for SMP-C
  - Main contribution of the paper
- Variables:
  - binary (as required), $\mu_{d,h}$ = 1 iff *d* is matched to hospital *h*
- Constraints:
  - Satisfied iff $\mu_{d,h}$ variables form a stable matching

Evaluation of SMP-C Encoding

- Via simulations
- Compared against Deferred Acceptance-style algorithms
- Scaled up to real-world sized markets (Psychology, not NRMP)
- Vastly outperformed DA-algorithms when percentage of couples was large

Evaluation of SMP-C Encoding

Many Couples, Failure Rate of DA-style



NB: SAT-E solves *every* problem.

Further Properties of SMP-C

- Joint work with A. Perrault and F. Bacchus
- Main goal: investigate strategy-proofness (truth telling) and resident-optimality in SMP-C mechanisms

Resident Pareto-Optimal Stable Matching

- In SMP, there is a relationship between female-optimal matchings and female-strategy-proofness. Does this relationship hold for SMP-C?
- No well-defined notion of “resident-optimal” stable matching
  - Use Resident Pareto-Optimal Stable Matching:
    - No stable match exists s.t. at least one resident is better off, and no residents are worse off

No resident strategy-proof stable mechanism exists

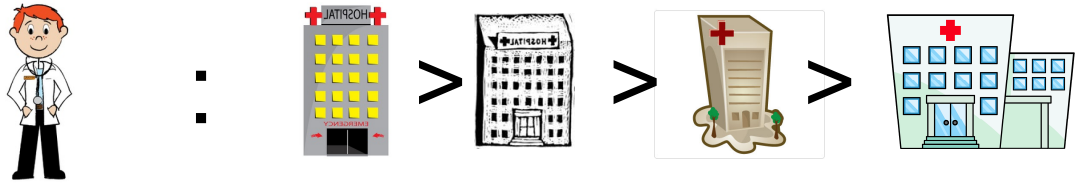
- Given any mechanism y() that returns a stable matching, there exists some set of resident and hospital preferences such that some resident can lie about their preferences and receive a more desirable match
- Proof: Found a counterexample using our SAT solver
  - Found an SMP-C problem instance with a unique stable matching that is manipulable

Some SMP-C problem instances not manipulable via truncations

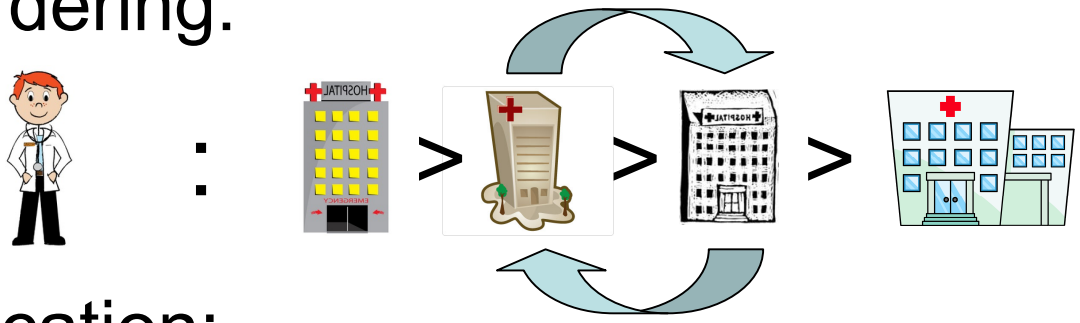
- Given some mechanism y() such that y() returns a resident Pareto optimal stable match, and some problem instance $\mathcal{I}$ such that $\mathcal{I}$ admits only a *unique* resident Pareto optimal match, then:
  - y() is manipulable ONLY via reorderings, not truncations

Some SMP-C problem instances are not manipulable via truncations

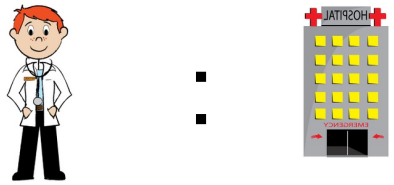
- Truthful preferences:



- Reordering:



- Truncation:



Resident Pareto Optimal Stable Mechanism for SMP-C

- Extension of the SAT encoding
  - Run the SAT solver multiple times, each time ensuring that any new solutions will be a Pareto improvement
    - Via adding additional constraints each time a solution is found
    - Resident Pareto improving match:
      - At least one resident has a more desirable match, no resident has a less desirable match

Conclusions

- Stable Matching Problems are ubiquitously used and techniques like Gale-Shapley are applied in large, real-world sized markets
- Very active research area

Thank you! Any Additional Questions?

Acknowledgements:

We acknowledge the support of NSERC. Drummond and Perrault additionally supported by OGS; Drummond additionally supported by Microsoft Research Graduate Women's Scholarship. We thank Tyler Lu and Iain McBride for generous use of their code; Ettore Damiano, David Manlove, and Péter Biró for helpful discussions; and the reviewers for their suggestions.