CSC200: Lecture 14

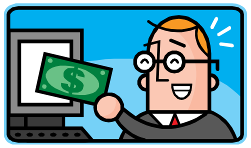
- Today
  - Recap: Second-price auctions
  - First-price auctions: Ch.9.5, 9.7A (ignore “General Distributions”)
- Next few lectures
  - Auctions: Chapter 9 (plus additional topics)
  - Matching markets: Ch.10

- Announcements
  - Quiz 3 this Friday. Do you prefer beginning or end of tutorial? The best way to study for the quiz is to review all the simple games in the slides and text and their Nash equilibria.
  - Assignment 2 is now available and due November 25.

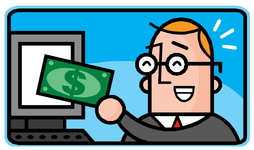
What is your “value” for an item?

- Let’s be precise about what your *value (or valuation)* for an item
- Think of it as the price at which you are *indifferent* between receiving the item at that price or not receiving the item at that price
  - no bargaining is allowed
  - no “outside” options to obtain the item (at a cheaper price)
  - it’s a one-time only, most you are willing to pay now

Recap: Truthful Bidding in Second-Price Auction

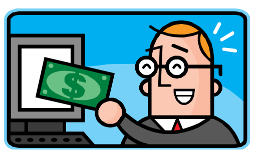


b_1 = \$125



v_2 = \$105

b_2 = ???

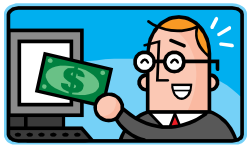


b_3 = \$90

b_4 = \$65

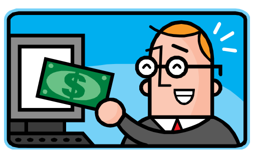
- Consider actions of bidder 2
  - Ignore values of other bidders, consider only their bids. Their values don't impact outcome, only bids do.
- What if bidder 2 bids:
  - truthfully \$105?
    - loses (payoff 0)
  - too high: \$120
    - loses (payoff 0)
  - too high: \$130
    - wins (payoff -20)
  - too low: \$70
    - loses (payoff 0)

Recap: Truthful Bidding in Second-Price Auction



b_1 = \$95

v_2 = \$105
b_2 = ???



b_3 = \$90

b_4 = \$65

- Consider actions of bidder 2
  - Ignore values of other bidders, consider only their bids. Their values don't impact outcome, only bids do.
- What if bidder 2 bids:
  - truthfully \$105?
    - wins (payoff 10)
  - too high: \$120
    - wins (payoff 10)
  - too low: \$98
    - wins (payoff 10)
  - too low: \$90
    - loses (payoff 0)

Equilibrium: Second-Price Auction Game

- Suppose k wins with truthful bid v_k
  - Notice k's payoff must be zero (if tied) or else positive
- Bidding b_k higher than v_k:
  - v_k already highest bid, so k still wins and still pays price p equal to second-highest bid $b_{(2)}$
- Bidding b_k lower than v_k:
  - If b_k remains higher than second-highest bid $b_{(2)}$ no change in winning status or price
  - If b_k falls below second-highest bid $b_{(2)}$, k now loses and is worse off (or at least no better if tied with $b_{(2)}$)

Equilibrium: Second-Price Auction Game

- Suppose k loses with truthful bid v_k
  - Notice k's payoff must be zero and highest bid $b_{(1)} > v_k$
- Bidding b_k lower than v_k:
  - v_k already a losing bid, so k still loses and gets payoff zero
- Bidding b_k higher than v_k:
  - If b_k remains lower than highest bid $b_{(1)}$, no change in winning status (k still loses)
  - If b_k is above highest bid $b_{(1)}$, k now wins, but pays price p equal to $b_{(1)} > v_k$ (payoff is negative since price is more than it's value)

- So bidding truthfully is a *dominant strategy*; optimal no matter what others are bidding

Other Properties: Second-Price Auction

- Elicits true values (payoffs) from players *(despite being unknown a priori)*
  - no bid shading: your bid influences whether you win or lose, *but has no impact on the price you pay if you win* (the critical fact that induces truthful bidding)
  - We'll see this again with *VCG mechanisms* (for more general auctions)
- Allocates item to bidder with highest value (maximizes social welfare)
- Surplus is divided between seller and winning buyer
  - splits based on second-highest bid (lowest price winner could reasonably expect)
- Outcome is similar to English auction (ascending auction)
  - raise prices gradually, bidders drop out until one remains
  - until price exceeds *k*'s value, *k* should stay in auction
    - drop out too soon: you lose when you might have won
    - drop out too late: will pay too much if you win
  - last bidder remaining has highest value, pays 2nd highest value!
  - games are in fact "strategically equivalent"; seller gets same price
    - with some "slop" due to bid increment in English auction

Recap: Single-item Auctions (Sell-side)

- Assume seller with one item for sale
- Several different formats
  - **Ascending-bid (open-cry) auctions** (aka *English auctions*)
    - price rises over time, bidders drop out when price exceeds their "comfort level"; final bidder left wins item at last drop-out price
  - **Descending-bid (open-cry) auctions** (aka *Dutch auctions*)
    - price drops over time, bidders indicate willingness to buy when price drops to their "comfort level"; first bidder to indicate willingness to buy wins at that price
  - **First-price (sealed bid) auctions**
    - bidders submit "private" bids; highest bidder wins, pays price s/he bid
  - **Second-price (sealed bid) auctions**
    - bidders submit "private" bids; highest bidder wins, pays price bid by the *second-highest bidder*

First-price Auction

- Bidders submit "private" bids; highest bidder wins, pays her bid
- 1st-price auction seems more intuitive than 2nd-price
  - but it makes bidders reason more *strategically*
  - in fact it is not obvious exactly how to determine your bid
  - question: why don't you bid your true value for the item?
    - *if you win, your payoff is zero (you pay exactly what it's worth, i.e., the price at which you're indifferent between taking the item of leaving it)*
    - *this suggests you want to "shade" your bid lower than true value; but by how much?*
- To better understand a first-price auction, let's formulate it as a game

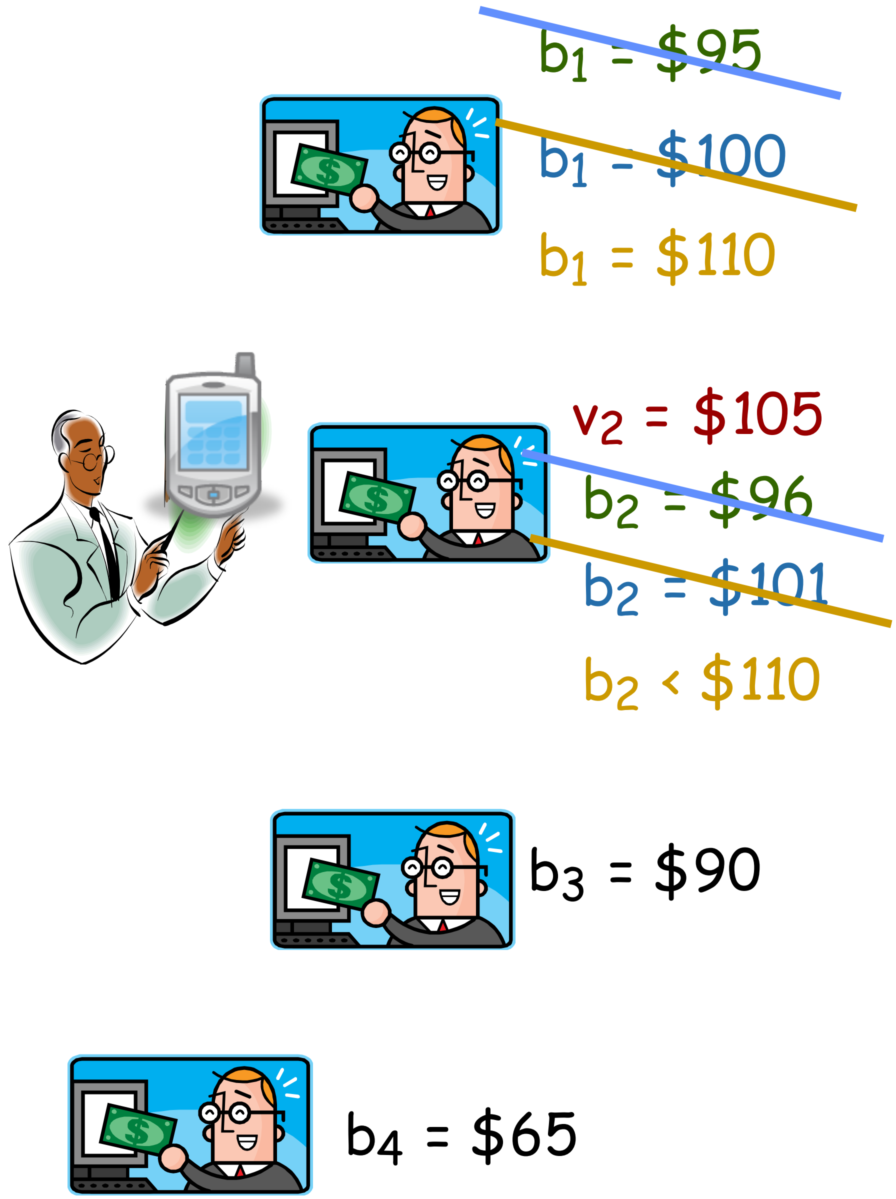
The First-Price Auction Game

- *n* players (bidders)
- each player *k* has value v_k for item
  - assume v_k between [0,1] (for concreteness only)
- strategies/actions for player *k*: any bid b_k between [0,1]
- outcomes: player *k* wins, pays price *p* (equal to her bid)
  - *more than n outcomes: outcome includes price paid by winner*
- payoff for player *k*:
  - if *k* loses: payoff is 0
  - if *k* wins, payoff depends on price *p*: payoff is $v_k - p$
- Notice: game is incomplete information (like 2nd-price)
  - *no player actually knows the payoffs of the other players*

First-Price Auction: No dominant strategy

- Claim: there is no dominant strategy for any player k
- Other players bid: let highest *"bid from others"* be $b_{(1)}$
  - If value v_k is greater than $b_{(1)}$ then k's *best bid* is b_k that is just a "shade" greater than $b_{(1)}$ (depends on how ties are broken)
  - This gives k a payoff of (just shade under) $v_k - b_{(1)} > 0$
  - If k bids less than $b_{(1)}$: k loses item (payoff 0)
  - If k bids "a lot" more than $b_{(1)}$: pays more than necessary (so k's payoff is reduced)
  - Notice k should never bid more than v_k
- So k's optimal bid *depends on what others do*
- Thus k needs some prediction of how others will bid
  - requires genuine *equilibrium analysis*

Bid Shading in First-Price Auction

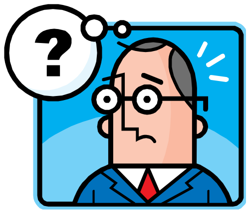


- Consider actions of bidder 2
  - ignore values of other bidders, consider only bids.
  - assume "bid increment" of $1 and ties are broken against bidder 2
- If bidder 1 bids $95:
  - bidder 2 *should bid* $96
    - wins with payoff 9
  - if 2 bids $94, loses (0)
  - if 2 bids $97, payoff 8
- If bidder 1 bids $100
  - bidder 2 *should bid* $101
    - wins (payoff 4)
- If bidder 1 bids $110
  - bidder 2 should bid "less"
    - loses (payoff 0)

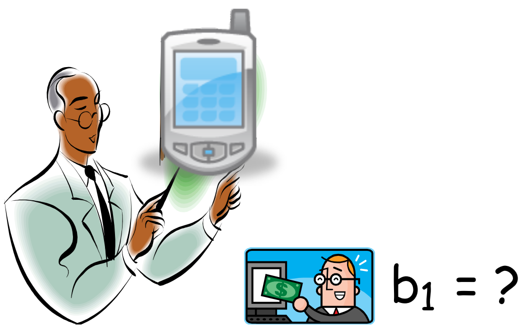
First-Price Auction: How much to shade?

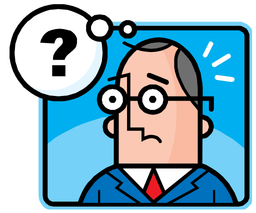
- Player *k* doesn't know values/bids of other bidders
  - cannot choose a strategy that is *guaranteed* to win or to maximize payoff
  - *k* needs to have some beliefs about how others will bid
  - *but how others will bid depends on their underlying values*
- Unlike earlier games, players need predictions (beliefs) about other players' *payoffs*, not just their strategies
  - we'll (sort of) formalize this in a few minutes
- Suppose bids of others are random between 0 and 1
  - *k* should shade its bid *more* (bid lower) when there are *fewer* competitors (the highest competing bid more likely to be lower)
  - … and should shade its bid *less* (bid higher) when there are *more* competitors

Bid Shading in First-Price Auction



What bid b_k should bidder k offer?





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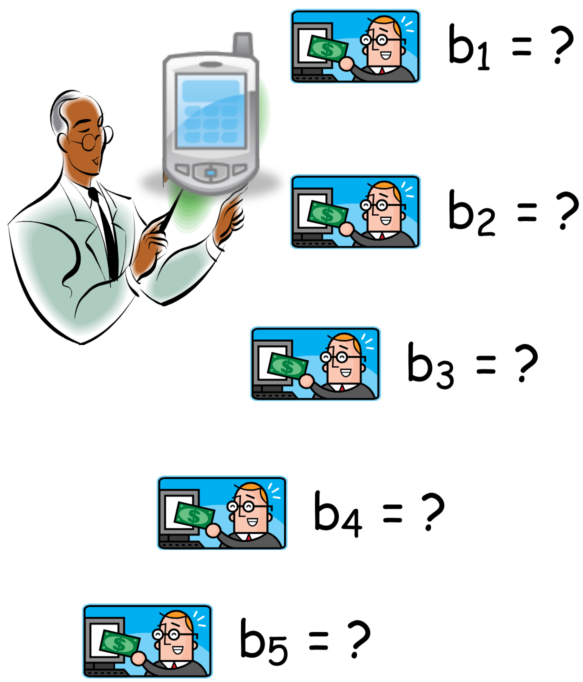

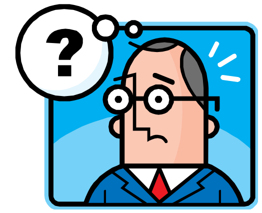

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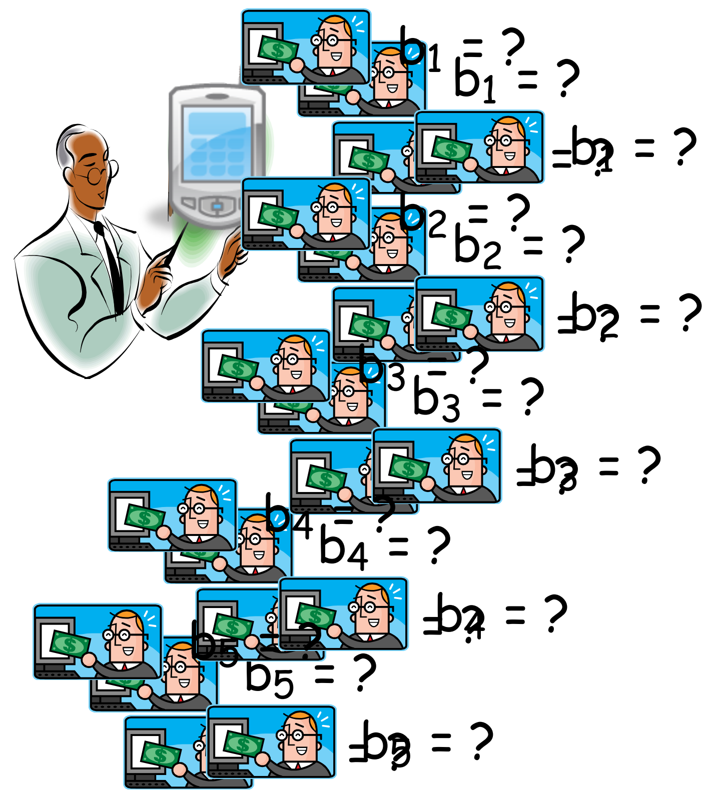


Equilibrium: First-Price Auction

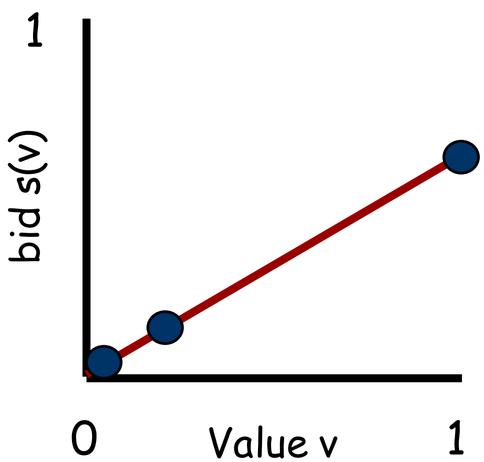
- We'll give a simple analysis
  - main point: take away *the flavor of the analysis* if not specifics
- **Game of incomplete information**
  - *k*'s *strategy s* depends on value v_k : $s_k(v_k)$ selects a bid b_k in [0,1]
    - critically, the selected bid can differ for different valuations v_k
    - critically, ***other players*** have strategies too: s_j
  - *k*'s payoff depends on its strategy and the strategy of others (as in Nash equilibrium), but also on *its value* and *the value of others*
    - *this is what distinguishes complete from incomplete info games*
- Let's look at game with *two bidders* *k* and *j*
  - Assume that their values are drawn randomly (*uniformly*) from the interval [0,1] and that they both know this
  - Let's see what strategies would be in equilibrium…

Equilibrium: First-Price Auction

- A *bidding strategy* for k is just a function $s_k(v_k) = b_k$
  - it tells you what bid to submit taking your value for the item as input
  - e.g., truthful strategy: $s(0)=0$; $s(0.1) = 0.1$; $s(1) = 1$; etc…
  - e.g., $s(v) = ½v$ says "bid half your value": $s(0)=0$; $s(0.1)=0.05$; $s(1) = 0.5$; …
- Some simplifying assumptions
  - strategy is *strictly increasing* (if value is higher, bid is also higher)
    - intuitively makes sense, but some sensible strategies might not
  - strategy is differentiable (no sharp bends, breaks)
    - makes analysis easier, but not a critical assumption in general
  - strategy *cannot bid higher than value:* $s(v) \leq v$
    - an obvious requirement for rational bidders
  - strategies are *symmetric*: k and j use same function, i.e., s_k same as s_j
    - not necessary: we derive only a symmetric equilibrium (nonsymmetric equilibria may also exist)

Our Assumptions on Bidding Strategies

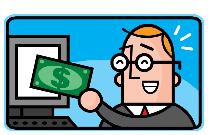
- Bidding strategy is increasing, differentiable, no greater than value, and symmetric



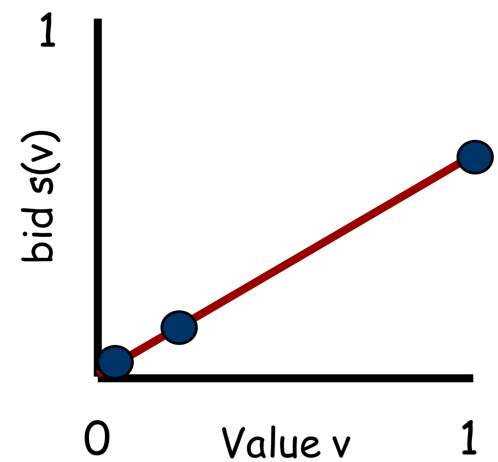
$v_k = 0 \longrightarrow b_k = 0$

$v_k = 0.2 \longrightarrow b_k = 0.1$

$v_k = 1 \longrightarrow b_k = 0.5$



Bidder k



$v_j = 0 \longrightarrow b_j = 0$

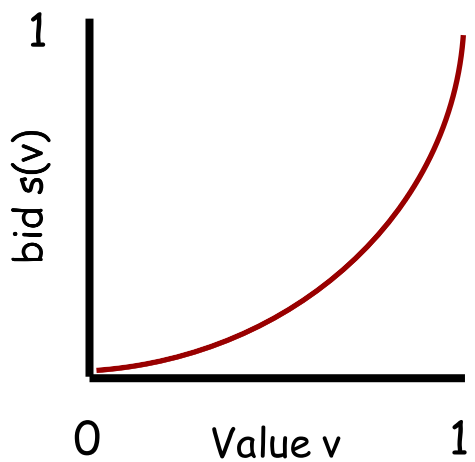
$v_j = 0.2 \longrightarrow b_j = 0.1$

$v_j = 1 \longrightarrow b_j = 0.5$

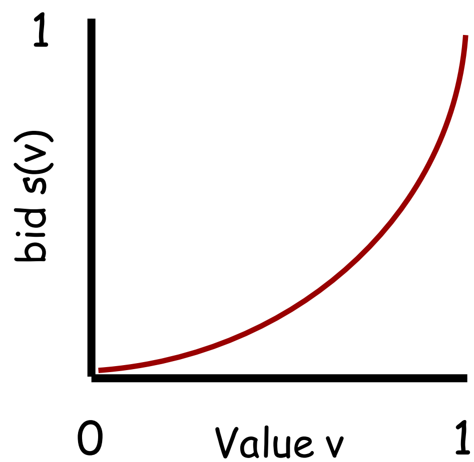
Bidder j

Our Assumptions on Bidding Strategies

- Bidding strategy is increasing, differentiable, no greater than value, and symmetric



Bidder k



Bidder j

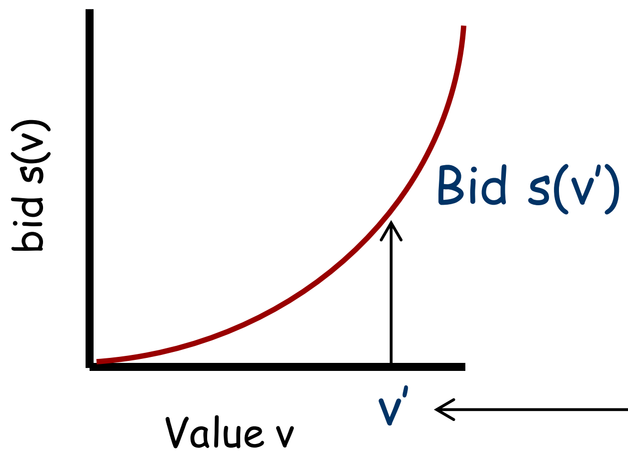
Equilibrium: First-Price Auction

- By symmetric assumption, *k* never wants to bid more than *s(1)* (since this is the maximum *j* will bid)
  - and obviously *s(0) = 0*, so it can't bid less than *s(0)*
- We want to find a strategy *s* such that neither *k* nor *j* deviate from *s*
- But for any strategy *s* satisfying our assumptions (specifically, need continuity), *k can produce any bid b_k between s(0) and s(1)* simply by plugging in some "pretend" valuation *v* (possibly different from its true v_k)

- Why do this? It restricts our search to strategies where the expected payoff for bidding $s(v_k)$, when *k*'s true value is v_k, is greater than the payoff for bidding *s(v)* for a different value *v* when *k*'s true value is v_k .
  - In other words, if *s(v)* is the best bid (better than $s(v_k)$) when *k*'s true value is v_k, just use a different strategy *s**, where $s^*(v_k) = s(v)$.

Fixing a strategy and changing the bid

- Even with a fixed strategy *s*, bidder *k* can produce any bid between *0* and *s(1)* by “pretending” to have a different value *v’* than his true *v*
  - … and it’s his bid that influences the outcome, not *s* per se



Plug in any value v’ you want (“lie to yourself”) to get any desired bid between 0 and s(1)

Bidder k

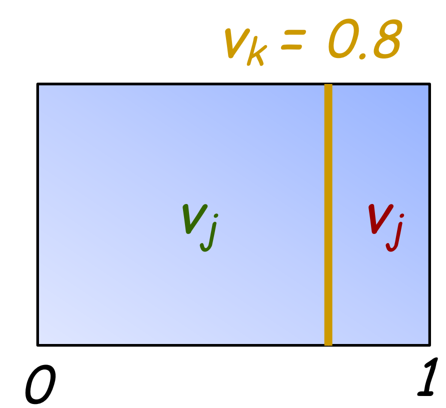
No need to “pretend”, just use a different function s where s*(v) = s(v’)*

What is expected value of strategy *s*?

- What is *k*'s expected payoff for playing *s*?
  - Payoff is zero if *k* loses
  - Payoff is "value minus bid" if *k* wins: $v_k - s(v_k)$
  - So if *k* wins with probability *p*, expected payoff is $p\,(v_k - s(v_k))$
- What is probability *k* wins?
  - Since strategies are symmetric, *k* wins just when $v_k > v_j$
  - This happens with probability v_k
  - So *k*'s expected payoff is $v_k(v_k - s(v_k))$

$Prob(v_j < 0.8) = 0.8$

$Prob(v_j > 0.8) = 0.2$



What is optimal bidding strategy?

- Want a strategy *s* where expected value of *bidding* $s(v_k)$ is better than bidding $s(v)$ for any other value v
  - If true value is v_k and bid is $s(v)$: probability of winning is v, and payoff if bidder wins is $v_k - s(v)$
  - So we want *s* satisfying: $v_k(v_k - s(v_k)) \geq v(v_k - s(v))$ for all v
- So payoff function $g(v) = v(v_k - s(v))$ should be maximized for the input v_k
  - we do this by finding derivative of *g*, setting the derivative to zero at point (input) v_k and solving a simple differential equation (see details in text if mathematically inclined)
- Result is: $s(v) = v/2$ (easy to *verify* that $s(v_k)$ is best bid for v_k)
- In other words, *the bidding strategy where both bidders bid <u>half of their value</u> is a Nash equilibrium*

Verification of Optimal Bidding Strategy

- Payoff function $g(v) = v(v_k - s(v))$ should be maximized by the input v_k
- Set: $s(v) = v/2$
- Then: $g(v) = v(v_k - v/2) = vv_k - ½v^2$
- Then: $g'(v) = v - v_k$
- Set $g'(v) = 0$ to obtain maximum payoff at $v = v_k$

For More Than Two Bidders

- Same analysis can be applied to uniform values: intuitive result
- If we have n bidders, a symmetric equilibrium strategy is for any bidder with value v_i to bid $(n-1)/n\ v_i$
  - e.g., if 2 bidders, bid half of your value
  - e.g, if 10 bidders, bid 9/10 of your value
  - e.g, if 100 bidders, bid 99% of your value
- Intuition (again): more competing bidders means that there is a greater chance for higher bids: so you sacrifice some payoff ($v_i - b_i$) to increase probability of winning in a more "competitive" situation

- Analysis is more involved for more general distributions of values
  - each specific form requires its own analysis, but general idea is similar to the uniform distribution case

Other Properties: First-Price Auction

- Bidders generally shade bids (as we've seen)
  - Does seller lose revenue compared to second-price auction? What do you think?
- *If bidders all use same (increasing) strategy,* item goes to bidder with highest value (this maximizes social welfare, like second-price)
  - but note that our symmetric equilibrium need not be only equilibrium!
- Outcome is similar to Dutch auction (descending auction)
  - lower prices until one bidder accepts the announced price
  - until price drops below *k*'s value, *k* should not accept it
    - jump in too soon: will pay more than necessary (equivalent to bid shading)
    - jump in too late: you lose when you might have won
  - first bidder jumping in pays the price she jumped in at (1^{st} price)
  - games are in fact "strategically equivalent"; seller gets same price
    - with some "slop" due to bid decrement in Dutch auction