CSC200: Lecture 12

- Today: some review and start of mechanism design
  - Pareto and Social Optimality (Ch.6.9)
  - Further Discussion of Braess' paradox (Ch.8.1-8.2)
  - Price of Anarchy
  - Mechanism design and start of auctions (starting Weds, Ch.9)
- Next few lectures
  - Auctions: Chapter 9 (plus additional topics)
  - Matching markets: Chapter 10
- Announcements
  - Pick up Quiz 1, Quiz 2 and Assignment 1 if you have not already done so.
  - Quiz 3 on Friday, Nov 13: game theory question
  - Office hours this week will be 3-4:30 (and not 2-3)
  - Grades can be seen on secure site with link from web page

Comparing Outcomes: Optimality

	Quiet	Confess
Quiet	-1 / -1	-10 / 0
Confess	0 / -10	-4 / -4

- We called some NE "undesirable"
  - e.g., Nash equilibrium in Prisoners' dilemma: makes both people worse off than "cooperative" outcome (Quiet/Quiet)
  - e.g., equilibrium in Braess's paradox: there is a way to route traffic that makes everyone better off (as we will soon see)
- Need some notion of "optimality" to compare payoff profiles or *strategy profiles*

Pareto Optimality

	Quiet	Confess
Quiet	-1 / -1	-10 / 0
Confess	0 / -10	-4 / -4

- *Pareto optimality:* a strategy profile is *Pareto optimal* iff no other strategy profile gives some player(s) a higher (expected) payoff without lowering the payoffs of others
  - Everyone would “agree” to *switch* from a non-Pareto optimal profile to a Pareto-optimal profile (if agreement “enforceable”)
- In Prisoners Dilemma: the unique NE is not Pareto optimal
  - Cooperative profile is better for all involved (but not *enforceable*)
  - Note: Q/C and C/Q are also both Pareto optimal (only Quiet player wants to switch)

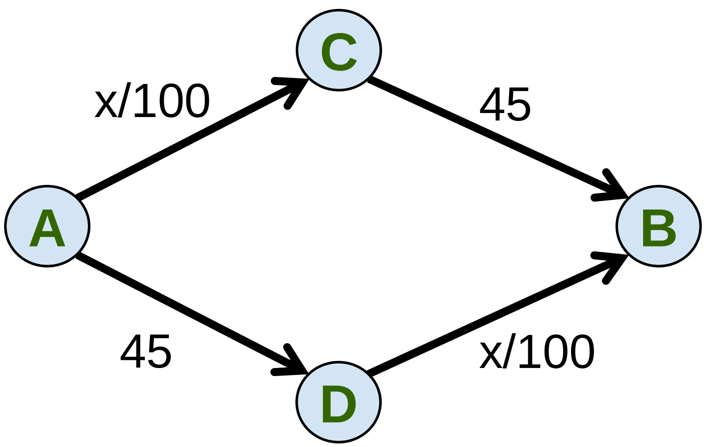
Social Optimality

	Quiet	Confess
Quiet	-1 / -1	-10 / 0
Confess	0 / -10	-4 / -4

- Pareto optimality desirable, but weak
  - and not always achievable in “noncooperative” games
  - cooperative game theory: models commitments/agreements
- *Social optimality:* a strategy profile is *socially optimal* iff it gives rise to the highest *sum* of expected payoffs
  - Total payoff to entire group (*social welfare*) is maximized
  - Assumes that summing payoffs is sensible
    - *Alternatives? Other notions of “fairness” or social utility?*
    - *Would everyone “agree” to this?*
  - If a profile is socially optimal, it must be Pareto optimal (if one player better off without hurting others, total payoff must increase), but not vice versa
- In PD: the unique NE is not Pareto (or socially) optimal

A Traffic Network (Recap)

- Let's look at the game with network structure in action space
  - *Stylized highway network:* travel time varies with traffic
  - if *x* cars on a link (segment) travel time is as labeled
    - varies on A—C and D—B but fixed on A—D and C—B

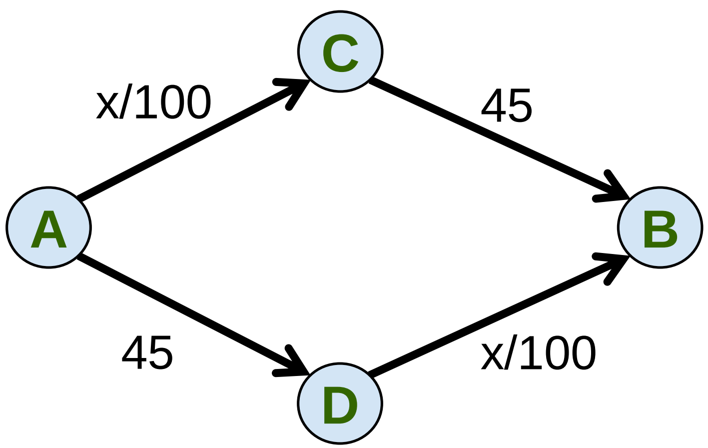


Example: Suppose 4000 drivers Must get from A to B each AM. And 3000 take route A-C-B, 1000 take A-D-B

- route C: 75 mins
- route D: 55 mins

Traffic Flow in Equilibrium (Recap)

- Suppose 4000 cars travel from A to B each morning
  - *What is equilibrium traffic flow?*
- Model as a game with 4000 players
  - each driver can choose route A—C—B or A—D—B
  - each driver desires minimum *personal* travel time: payoff is *-mins*



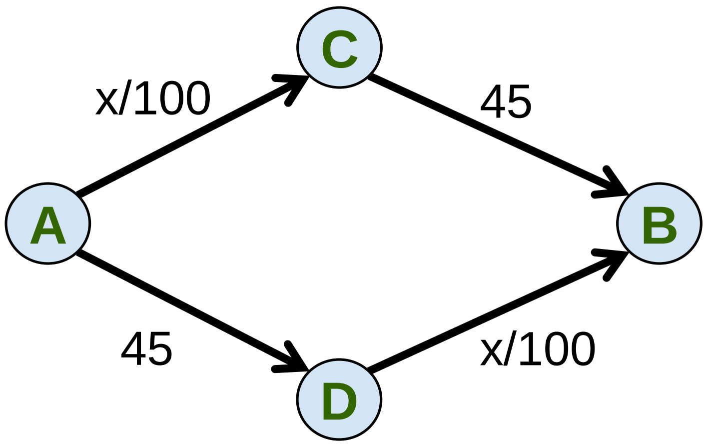
Many Nash equilibria!

But all are "equivalent":
- 2000 drivers take C
- 2000 drivers take D
- all travel times: 65mins

Traffic Flow in Equilibrium (Recap)

- (Any) profile <*2000 C, 2000 D*> is a NE
  - each route is equally fast: 65 mins, no incentive to switch
  - in fact, if a driver switches (e.g., from C to D): her travel time goes up from 65 mins to 65.01 mins
  - How many NE? $\binom{4000}{2000} \approx 1.6 \times 10^{1202}$

- Why is <*n C, 4000-n D*> not a NE if $n \neq 2000$?
  - Any driver on slower route will want to switch to faster route



Social Optimality

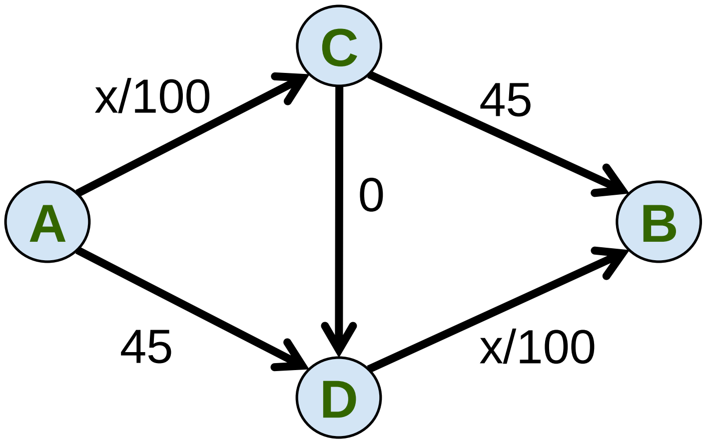
- The Nash equilibrium (we'll speak as if there's one, since they are all interchangeable) is in fact *socially optimal* and is the only socially optimal way to arrange traffic
  - it minimizes the population's total (equiv. average) commute time
  - in the NE (2000/2000): everyone has 65 minute commute time
  - if you shift balance to (2001 C, 1999 D):
    - *1999* drivers see commute time drop by 0.01 (64.99 mins)
    - but ***2001*** see commute time rise by 0.01 (65.01 mins)
    - total commute time goes up by 0.02 mins
  - (2100 C, 1900D): total increase of 200 mins
  - (3000 C, 1000D): total increase of 20,000 mins (about 2 weeks)
  - (4000 C, 0D): total increase of 80,000 mins (almost 2 months)

How might NE emerge in practice?

- With 10^{1200} NE: ultimate equilibrium selection problem!
  - 4000 drivers didn't call each other up this AM and coordinate
  - Iterative process? Try out a route… if it's fast you stick, if it's slow you switch?
    - Suppose 4000 start with C: what do they do the next day?
    - All switch to D! Then all switch back to C, …
  - More likely, a probabilistic process, some people more amenable to switching than others… and the slower it is the more likely you are to switch… over time after a process of gradual adjustment leads to something that is approximate NE
- What's nice about this: self-organization based on self-interest makes everyone better off, indeed *as well-off as possible*, since it maximizes social welfare.

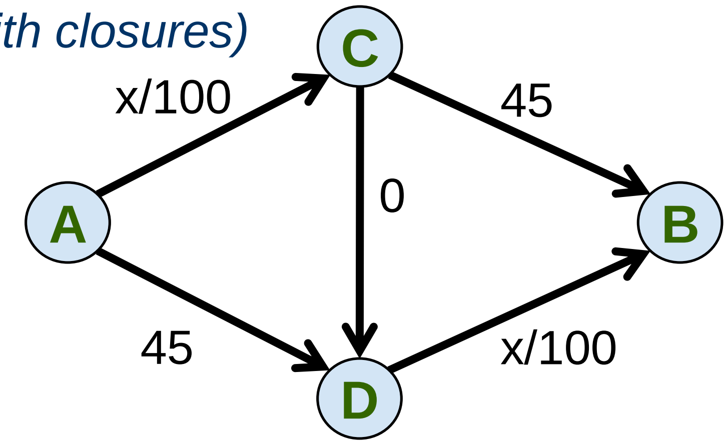
Braess' Paradox

- Our new mayor treats Torontonians to reward them for her/his recent victory
  - adds a new superhighway to reduce everyone's commute time
  - link is *much* faster (we'll say time=0, but any "small" value works)
  - what happens to traffic patterns?



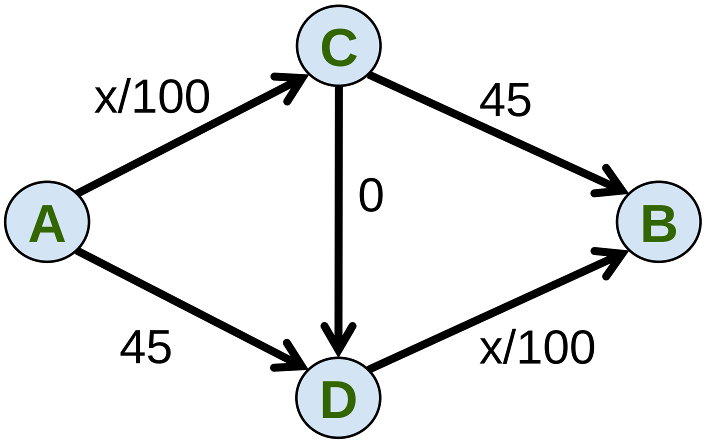
Braess' Paradox

- Unique NE in new game:
  - *everyone* takes A-C-D-E; commute increases from 65 to 80 mins!
- Why?
  - the *longest time* that A-C or D-B can take: 40 mins (all 4000 drivers)
  - so A-C-D *always* faster than A-D;   C-D-B *always* faster than C-B
  - so A-C-D-B is *dominant* for every driver
- "Paradox": adding capacity slowed everyone down
  - named for discover (Diettrich Braess, 1968)
  - *observed in some real traffic situations (with closures)*



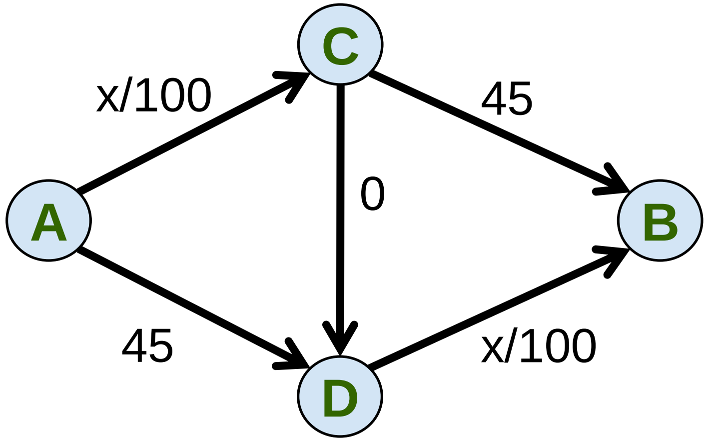
Why does it happen?

- Before new link:
  - all routes from A-B required one 45 min link
  - but *two* of them, so traffic split, easing congestion on A-C and D-B
- After new link:
  - everyone can avoid 45 min link
  - but *only one way to do so*: draws all traffic through C-D
  - leaves both 45 min links (A-D, C-B) unused!



How useful could the new link be?

- What is a *socially optimal* solution?
  - 2250 take A-C; 1750 take A-D;
  - 500 take C-D
  - 1750 take C-B; 2250 take D-B
  - *reasoning: find split of 4000 drivers over links A-C, A-D that minimizes their average time to C or D (2250/1750 split, minimizes quadratic function)*
  - *by symmetry, 2250/1750 on D-B, C-B minimizes avg. time getting 4000 drivers from C or D to B*
  - *but we can swap 500 people from C-D for free to achieve both*
- Socially optimal solution can only be "imposed"
  - will not emerge in equilibrium
  - nobody will be willingly take A-D *or* C-B



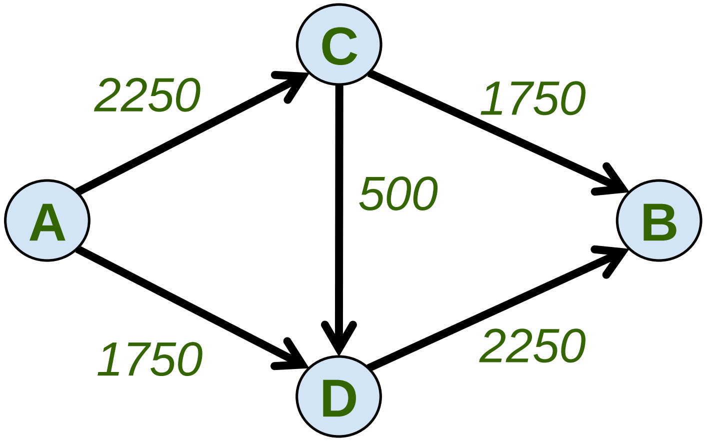
Minimizing Total Traffic Time

- For those interested in how to arrange flow:
  - Goal: arrange traffic to minimize total commute time from A-B
  - Subgoal: minimize time from A to "C or D" to split traffic flow between A-C and A-D; by symmetry, can use analogous arrangement to get from "C or D" to B.
  - So: send g drivers on A-C and $(4000 - g)$ drivers on A-D
  - Total time is: $g \cdot \frac{g}{100} + (4000 - g) \cdot 45 \; = \; 0.01g^2 - 45g + 180000$
  - Minimize by setting derivative to zero: $0.02g - 45 = 0$
  - Solving gives $g = 2250$, so send 2250 on A-C, 1750 on A-D

How would you achieve social optimum?

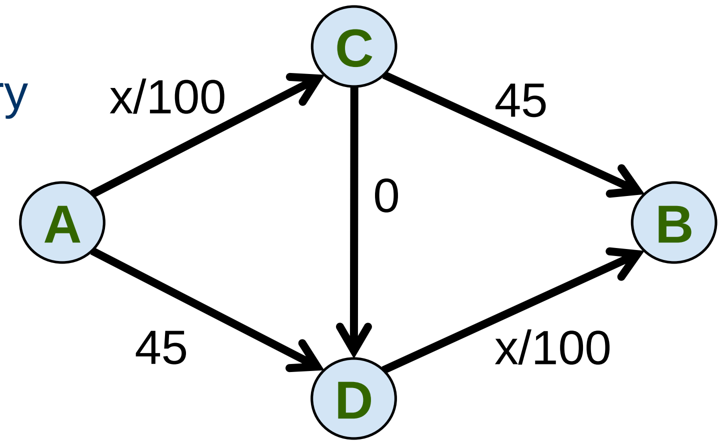
- Social optimum:
  - 2250 take A-C; 1750 take A-D;
  - 500 take C-D
  - 1750 take C-B; 2250 take D-B
- Socially optimal solution can only be “imposed”
  - will not emerge in equilibrium
  - nobody will willingly take A-D *or* C-B
- Gov’t/policy makers must somehow:
  - Limit capacity on A-C (max 2250)
  - Limit capacity on C-D (max 500)
  - How: licenses? tolls?
    subsidies? need? other?

Socially optimal driver distribution



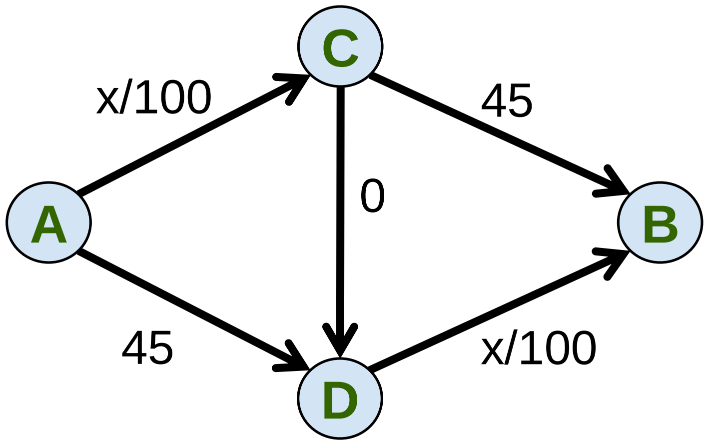
Value of new link (if socially optimal traffic flow)

- Compare new link with ***imposed*** social optimum to *no link*
  - *Without new link: everyone takes 65 mins*
  - With new link, social optimum average commute time is 64.6875 mins
    - 500 have A-C-D-B (45 mins)
    - 3500 have ADB or ACB (67.5 mins)
  - Total time saved: 1250 mins (avg. of 18.75 sec per driver)
  - Not Pareto improving: 500 people save a lot at expense of 3500 others
    - Which 3500 will put up with it?
    - Which 500 should be allowed to benefit?
- Is the new link worth it?
  - Even with imposed optimum, it saves very little time on average (19s. per driver) compared to optimum without link (which is NE, requires no rules)
  - Imposed optimum: unequal treatment



Value of Imposing Social Optimum

- Assume new link exists: if you *could* impose social optimum, how much better off is "society" compared to NE
  - Average per person drops to 64.6875 mins vs. 65 mins (total time savings of over 6 person-weeks every day)
  - So allowing people to act in their own interests (equilibrium) causes society (and in this case, every member of society) to suffer (aka *Tragedy of the Commons*)



Price of Anarchy

- Computer scientists have studied the following question:
  - consider a game (usually of a specific form) and ask what social cost we can derive from the *socially optimal profile* (OPT)
  - consider *social cost of the (worst or best) NE* of the same game (SCNE); in network congestion example, unique NE so worst = best
  - what is the ratio OPT:SCNE?
- In other words, how much societal benefit do you sacrifice by letting everyone choose their own actions?
  - sometimes called the ***price of anarchy (when considering worst NE)***
- For networks with *linear* cost functions (and *non-atomic routing*):
  - known that one can increase optimal social cost *by no more than a factor of 4/3* (or that OPT:SCNE is no more than 3/4)
  - also known that by adding a link, the change in the SCNE is at most 2 (i.e., new NE with new link causes avg. time to increase by at most twice relative to the old NE in the network prior to adding the link)
- Read 8.3 (Advanced) for more technical discussion